Noncommutative Hypersurfaces and Support for (Hopf) algebras (w/ J. Pevtsova)

k = a field.

Pt 1: Noncommutative complete intersections (& cohomology)

Pt 2: "Applications" to support theory.

– Complete intersections

A (local) complete intersection is an alg R w/ a presentation

regular (local)

$R = Q/(f_1, \ldots, f_n)$ where the f_i's form a regular sequence

Rephrase: Can think of this as a triple (Z, Q, R)

$Z = P[f_1, \ldots, f_n]$, $Q = Z$-alg

$R \cong k \otimes_Z Q$

and Q is sufficiently flat over Z

$\mathrm{Tor}^Z_{>0}(Q, k) = 0$.

Koszul complex is exact away from deg 0.

Ex: $R = \dfrac{k[x_1, \ldots, x_n]}{(x_i^{l_i})}$

$Q = k[x_1, \ldots, x_n]$

$Z = k[x_1^{l_1}, \ldots, x_n^{l_n}]$.

- Want to think about "noncommutative complete int" (ncci), and ask what's the utility of such a notion.

– Cohomology for CCI.

Theorem (Gulliksen 70's): If R is a commutative complete intersection, $\dim(R) < \infty$, then

(i) For any fin gen'd R-module M

{ an endo ring of an obj in a fancy category } $= \mathrm{Ext}^*_R(M, M) \curvearrowleft$ finitely gen'd alg.

(ii) For any fin gen'd M and N
$$\mathrm{Ext}^{\bullet}_R(M,N)$$
is a fin gen'd module over $\mathrm{Ext}^{\bullet}_R(M,M)$ and $\mathrm{Ext}^{\bullet}_R(N,N)$ independently.

How do we see this (Gulliksen, Eisenbud, Gasharov, Peeva, etc.)

The map $Q \to R$ provides an action $Z \subseteq$

"Alg of cohom operators" $A_Z = k[\chi_1, \dots, \chi_n] \curvearrowright D^b(R)$

χ_i deg 2; χ_i dual to the f_i.

- $\forall M$ have $A_Z \xrightarrow[\text{alg map}]{\text{central}} \mathrm{Ext}^{\bullet}_R(M,M)$ — Prove that finite alg map
- Two actions of A_Z on $\mathrm{Ext}^{\bullet}_R(M,N)$ agree. — Prove that fin gen'd A_Z-mod.

– Deformations and NCCIs

Def^n: A deformation of a (noncomm) alg R is a pair (Z, R) where Z = augmented comm and Q = flat Z-alg w/ proj $Q \to R$ s.t. $k \otimes_Z Q \cong R$.

Call a deformation (formally) smooth if Z is (formally) smooth $/k$.

Def^{n}: A noncomm complete intersection is a finite-dimensional alg R w/ (Z, Q) a smooth deformation w/ Q Noetherian and of finite global dimension.
"nc regular"

This definition is built for Hopf algebras.

Fake defn: A Hopf alg = any algebra w/ the word "group" in its name.

Finite group scheme = a affine group scheme G s.t. $\mathcal{O}(G)$ are fin dim'l algebra.

Ex: G a smooth alg group $/\mathbb{F}_p$

$$\mathcal{O}(G_{(r)}) = \text{Frobenius kernel} = \mathcal{O}(G)/(f^{p^r} : f \in \mathfrak{m}_1)$$

$$\mathcal{O}(GL_{n(r)}) = k[x_{ij}]\det^{-1}/(x_{ij}^{p^r} - \delta_{ij}).$$

Examples: $/\mathbb{F}_p$. For G a fin group scheme, can embed G into smooth $\mathcal{H}$, $G \hookrightarrow \mathcal{H}$,

$$Z = \mathcal{O}(\mathcal{H}/G) \subseteq \mathcal{O}(\mathcal{H}) \to \mathcal{O}(G)$$

realizes $\mathcal{O}(G)$ as ncci.

$/\mathbb{F}_p$. Take $\mathfrak{g}$ a restricted Lie alg, $(-)^{[p]} : \mathfrak{g} \to \mathfrak{g}$,

$$Z = k[x^p - x^{[p]} : x \in \mathfrak{g}] \subseteq \mathcal{U}(\mathfrak{g}) \longrightarrow u^{res}(\mathfrak{g}) = \mathcal{U}(\mathfrak{g})/(x^p - x^{[p]} : x \in \mathfrak{g})$$

realizes $u^{res}(\mathfrak{g})$ as ncci.

$/\mathbb{C}$. Consider $q \in \mathbb{C}^\times$, $\operatorname{ord}(q) = l$, $[a_{ij}]$ skew symm matrix

$$\mathfrak{a}_q = \mathbb{C}_q[x_1, \dots, x_n]/(x_1^l, \dots, x_n^l) \qquad \mathbb{C}_q[x_1, \dots, x_n] = \mathbb{C}\langle x_1, \dots, x_n \rangle$$

$Z = \mathbb{C}[x_1^{\ell}, \ldots, x_n^{\ell}] \subseteq \mathbb{C}_q[x_1, \ldots, x_n] \to A_q \qquad (x_i x_j - q^{a_{ij}} x_j x_i)$

is ncci.

/$\mathbb{C}$. $U^{res}(\mathfrak{g})$ admits char 0 analog def by semisimple of $q \in \mathbb{C}^\times$ w/ ord$(q) = p$. $U_q(\mathfrak{g})$.

$Z \subseteq U_q^{DK}(\mathfrak{g}) \to u_q(\mathfrak{g})$ NCCI.

char 0 analog of $U(\mathfrak{g})$

- Chem

Theorem [N-Pevtsova]: If R is (fin-dim) ncci

$\Rightarrow$

- $\mathrm{Ext}^{\bullet}_R(M,M)$, M fin gen'd, is fin gen'd alg
- $\mathrm{Ext}^{\bullet}_R(M,N)$ is fin gen'd mod / $\mathrm{Ext}^{\bullet}_R(M,M)$.

- Hypersurfaces and support
  - Tensor categories and "support"
  - Hypersurface support
  - Some theorems [N8].

– Hopf alg's and $\otimes$-categories

has duality $V \rightsquigarrow V^*$

Def^n: A Hopf alg u is an alg w/ a rigid monoidal structure $\otimes$ on $rep(u) = \{$fin dim u-modules$\}$.

(fin-dim) Hopf algebras $u \rightsquigarrow$ (finite) $\otimes$-categories.

Ex $\quad kG \longmapsto rep(G)$

$u^{res}(\mathfrak{g}) \rightsquigarrow$ restricted rep's of $\mathfrak{g}$.

Theorem [Happel, Larson-Sweedler]: If u a fin-dim'l Hopf alg $\Rightarrow$ $rep(u)$ is a Frobenius category ($Proj = Inj$), and $proj(u)$ is a $\otimes$-ideal in $rep(u)$. So can form the stable category

$$stab(u) = rep(u)/proj(u) = D^b(u)/\langle proj(u)\rangle$$

$\rightsquigarrow$ $\otimes$-triangulated category.

fin-dim Hopf alg $u \rightsquigarrow stab(u)$ $\otimes$-triangulated cat.

– Thick ideals and support for $\otimes$-Δ.

Let $\mathcal{C}$ be a $\otimes$-triangulated category.

Def^n: A thick ideal in $\mathcal{C}$ is a thick subcategory $\mathcal{K} \subseteq \mathcal{C}$ s.t. $\mathcal{C} \otimes \mathcal{K} \subseteq \mathcal{K}$ and $\mathcal{K} \otimes \mathcal{C} \subseteq \mathcal{K}$.

A support theory for $\mathcal{C}$ is a pair $(Y, supp)$ of a top space Y and an assignment

$$supp : \{\text{object in } \mathcal{C}\} \longrightarrow \{\text{Closed subsets in } Y\}.$$

Def^n: Call $(Y, supp)$ reasonable if

$\mathrm{supp}(V \otimes W) \subseteq (\mathrm{supp}(V) \cap \mathrm{supp}(W))$.

If (Y, supp) is reasonable $\Rightarrow$ get maps

$$(*) \quad \left\{ \begin{array}{c} \text{thick ideals} \\ \text{in } \mathcal{C} \end{array} \right\} \underset{K_?}{\overset{\mathrm{supp}}{\rightleftarrows}} \left\{ \begin{array}{c} \text{specialization} \\ \text{closed subset in } Y \end{array} \right\}$$

$$\mathcal{K} \longmapsto \mathrm{supp}(\mathcal{K}) = \bigcup_{V \in \mathcal{K}} \mathrm{supp}(V)$$

$$K_\Theta = \{V : \mathrm{supp}(V) \subseteq \Theta\} \longleftarrow\!\shortmid\ \Theta$$

[Ideas from: Rickard, Friedlander-Pevtsova, Balmer]

Def^n: Say (Y, supp) reasonable supp theory classifies ideals in $\mathcal{C}$ if $(*)$ are a bijection.

$\Rightarrow$ The global structure of $\mathcal{C}$ is completely encapsulated in the geometry of Y. "$Y = \mathrm{Spec}(\mathcal{C})$".

$\boxed{\text{Goal: Use ncci intuition + comm alg theory to produce a supp theory } \mathrm{supp}^{hyp} \text{ which classifies ideals in stab ncci.}}$

— Integrable Hopf algs

Def^n: Call a finite-dim'l Hopf alg u integrable if u admits smooth defo $\mathcal{U} \to u$ w/ $\mathcal{U}$ Noeth Hopf alg of fin gl dim, and $\mathcal{Z} \subseteq \mathcal{U}$ param central subalg a [Hopf subalg] in $\mathcal{U}$. [integrable u = Hopf ncci]

Example: • $\mathcal{U}(\mathfrak{g}) \to u^{res}(\mathfrak{g})$. • $\mathcal{O}(\mathcal{G}) \to \mathcal{O}(G)$

• $\mathbb{C}_q[x_1, \ldots, x_n] \to \mathfrak{a}_q$ • $\mathcal{U}_q^{DK}(\mathfrak{g}) \to u_q(\mathfrak{g})$.

Call $\mathcal{U} \to u$ in this case an integration of u.

Want to classify thick ideals in $\mathrm{stab}(\mathfrak{u})$ for $\mathfrak{u}$ w/ chosen integration $\mathcal{U} \to \mathfrak{u}$.

- Hypersurface support (fix $\mathcal{U} \to \mathfrak{u}$ some integration)

[following Avramov-Buchweitz '00]

Have $Z \subseteq \mathcal{U}$, and can consider any point

$$c : \mathrm{Spec}(K) \longrightarrow \mathbb{P}(m_Z/m_Z^2).$$

To this point associate a hypersurface alg

$$\mathcal{U}_c = \mathcal{U}_K/(f) \qquad [\mathcal{U}_K = \mathcal{U} \otimes K]$$

where $f \in m_{Z_K}$ w/ $\bar{f} \in (m_Z/m_Z^2)_K$ representing c.

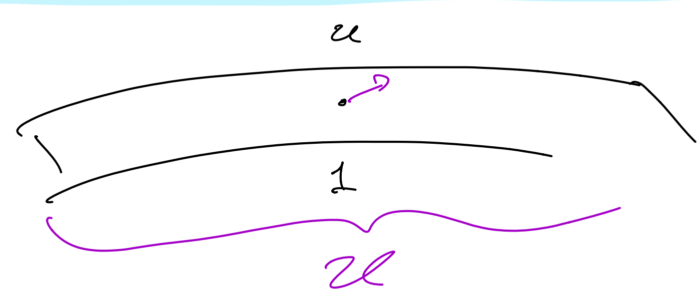
Consider $\mathcal{U}_c \to \mathfrak{u}$, say a $\mathfrak{u}$-rep V is supported at c if

$$\mathrm{proj\,dim}_{\mathcal{U}_c}(V_K) = \infty.$$

Def^{n} (N-Pev, following Av-Buch) Hypersurface support of $V \in \mathrm{stab}(\mathfrak{u})$ as

$$\mathrm{supp}^{hyp}(V) = \left\{ \begin{array}{c} \text{image of all} \\ c : \mathrm{Spec}(K) \to \mathbb{P}(m_Z/m_Z^2) \\ \text{s.t. } V \text{ supported at } c. \end{array} \right\}$$

$\mathrm{Spec}(Z)$



Theorem [N-Pev] • This definition depends on points $c \in \mathbb{P}(m_Z/m_Z^2)$, not representatives.

• $\mathrm{supp}^{hyp}(V) \subseteq \mathrm{supp}^{hyp}(1) =: Y$ for all V in $\mathrm{stab}(\mathfrak{u})$.

- $\mathrm{supp}^{hyp}(V) = \emptyset$ iff $V = 0$ in $\mathrm{stab}(\mathcal{C})$.

$\Longrightarrow$ Get nice support theory (Y, supp^{hyp}) on $\mathrm{stab}(\mathcal{C})$ integrable.

$\mathrm{Hom}(V, W) = \mathrm{Hom}(\mathbb{1}, V \otimes W^*)$

— Thick ideals

Theorem [NP]: (Y, supp^{hyp}) classifies thick ideals in $\mathrm{stab}(\mathcal{C})$ for the following examples:

$/\mathbb{C}$ • Quantum $\mathfrak{CI}$ $\mathcal{C}_q$, in which case $Y = \mathbb{P}^{n-1}$.

$/\mathbb{C}$ • Quantum borel $u_q(\mathfrak{b})$ in type A, $(\mathfrak{b} \leq \mathfrak{sl}_n)$, $Y = \mathbb{P}(\mathfrak{n})$.

$/\mathbb{F}_p$ • [Drinfeld doubles of some some solvable group schemes].

$/\mathbb{F}_p$ • $\mathcal{O}(G)$ for any fin grp scheme G, in which case $Y = \mathrm{Proj}(\mathrm{Ext}_{\mathcal{O}}(\mathbb{1}, \mathbb{1}))$ ~~ad~~ $\pi_0(G)$.