Tropical Ideals

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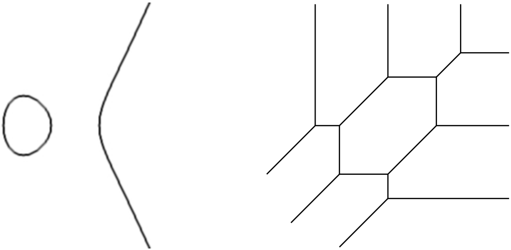
Goal: Introduce a scheme theory for tropical geometry.
Commutative algebra of ideals in the semiring of tropical polynomials.

Slogan: Tropical geometry is a combinatorial shadow of algebraic geometry.



Distraction question

If $I \subseteq S := K[x_0, \ldots, x_n]$ is a homogeneous ideal of (projective) dimension zero, then

$$\deg(I) = \sum_{p \in V(I)} \mathrm{mult}_p(V(I))$$

where the multiplicity is the length of $(S/I)_P$ as an S_P module, where P is the ideal of the point p.

Question: What proofs do you know that do *not* involve localization/primary decomposition?

The tropical semiring

$$\overline{\mathbb{R}} = (\mathbb{R} \cup \{\infty\}, \oplus, \odot),$$

where $\oplus = \min$ and $\odot = +$.

Examples: $5 \oplus 8 = 5$
$3 \odot 8 = 11$
$3 \odot (5 \oplus 8) = 8 = 3 \odot 5 \oplus 3 \odot 8$.
$(6 \odot 5) \oplus 10 = ?$

This is commutative, associative, ∞ is the additive identity, and 0 is the multiplicative identity.

Warning: No subtraction (so we have a semiring).

Semiring of tropical polynomials

The semiring of tropical polynomials is $\overline{\mathbb{R}}[x_1, \ldots, x_n]$.
$3 \odot x_1^2 \oplus 5 \odot x_1 \odot x_2 \oplus 7 = \min(2x_1 + 3, x_1 + x_2 + 5, 7)$.

Polynomials not functions! $x^2 \oplus x \oplus 0 \neq x^2 \oplus 0$.

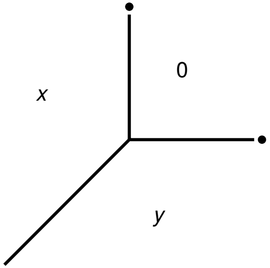
An ideal $I \subseteq \overline{\mathbb{R}}[x_1, \ldots, x_n]$ is a set closed under addition, and under (tropical) multiplication by elements of $\overline{\mathbb{R}}[x_1, \ldots, x_n]$.

Warning: Not always finitely generated!
Example: $\langle x \oplus y, x^2 \oplus y^2, x^3 \oplus y^3, \ldots \rangle$.

The tropical hypersurface $V(f)$ of $f \in \overline{\mathbb{R}}[x_1, \ldots, x_n]$ is

$$\{\mathbf{w} \in \overline{\mathbb{R}}^n : f(\mathbf{w}) = \infty \text{ or the minimum in } f(\mathbf{w}) \text{ is achieved at least twice}\}.$$

Example: $f = x \oplus y \oplus 0 = \min(x, y, 0)$



The variety of I is

$$V(I) = \bigcap_{f \in I} V(f).$$

Back to your first algebra class ...

When R is a semiring, the image of a semiring homomorphism is the quotient by a congruence.
This is an equivalence relation on R compatible with addition and multiplication: $a \sim b$ implies $a \oplus c \sim b \oplus c$, and $a \odot c \sim b \odot c$.

Giansiracusa bend congruence: For $f = \bigoplus c_{\mathbf{v}} \odot \mathbf{x}^{\mathbf{v}} \in \overline{\mathbb{R}}[x_1, \ldots, x_n]$, set $f_{\hat{\mathbf{u}}} = \bigoplus_{\mathbf{v} \neq \mathbf{u}} c_{\mathbf{v}} \odot \mathbf{x}^{\mathbf{v}}$. The bend congruence is

$$\mathcal{B}(I) = \{f \sim f_{\hat{u}} : f \in I, \mathbf{x}^{\mathbf{u}} \text{ is a monomial occurring in } f\}.$$

Example: $I = \langle x \oplus y \oplus 0 \rangle$. Then $B(I) = \langle x \oplus y \oplus 0 \sim x \oplus y \sim x \oplus 0 \sim y \oplus 0, \ldots \rangle$.

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Why this definition? For $I \subseteq K[x_1, \ldots, x_n]$, the "$K$-valued points" of $V(I)$ are $\mathrm{Hom}(K[x_1, \ldots, x_n]/I, K)$. When $K = \overline{K}$ this is in bijection with the closed points of $V(I)$: $\phi \mapsto (\phi(x_1), \ldots, \phi(x_n))$

This bijection holds tropically:
$\mathrm{Hom}(\mathbb{R}[x_1, \ldots, x_n]/\mathcal{B}(I), \overline{\mathbb{R}}) \leftrightarrow V(I)$
$\phi(x_i) = w_i$ is well-defined if and only if $f(\mathbf{w}) = g(\mathbf{w})$ for all $f \sim g \in \mathcal{B}(I)$, so if and only if $f(\mathbf{w}) = f_{\hat{\mathbf{u}}}(\mathbf{w})$ for all $f \in I, \mathbf{u}$.

Connection to usual algebraic geometry

Let K be a field with a valuation $\mathrm{val}\colon K \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ satisfying

1. $\mathrm{val}(ab) = \mathrm{val}(a) + \mathrm{val}(b)$ for all $a, b \in K$,
2. $\mathrm{val}(a + b) \geq \min(\mathrm{val}(a), \mathrm{val}(b))$ for all $a, b \in K$, and
3. $\mathrm{val}(a) = \infty$ if and only if $a = 0$.

Example: (trivial valuation) Any K, $\mathrm{val}(a) = 0$ for all $a \neq 0$.
Given $f \in K[x_1, \ldots, x_n]$, $f = \sum c_{\mathbf{u}} \mathbf{x}^{\mathbf{u}}$,

$$\mathrm{trop}(f) = \bigoplus \mathrm{val}(c_{\mathbf{u}}) \odot \mathbf{x}^{\mathbf{u}} = \min(\mathrm{val}(c_{\mathbf{u}}) + \mathbf{x} \cdot \mathbf{u}).$$

Examples:

- $f = x + y + 1$ $\mathrm{trop}(f) = x \oplus y \oplus 0$
- $(\mathbb{Q}, \mathrm{val}_2)$. $f = 2x^2 + 3xy + 4y^2 + 5x + 7y + 8$
  $\mathrm{trop}(f) = 1 \odot x^2 \oplus xy \oplus 2 \odot y^2 \oplus x \oplus y \oplus 3$.

Tropicalization

Given $f \in K[x_1, \ldots, x_n]$, $f = \sum c_{\mathbf{u}} \mathbf{x}^{\mathbf{u}}$,

$$\text{trop}(f) = \bigoplus \text{val}(c_{\mathbf{u}}) \odot \mathbf{x}^{\mathbf{u}} = \min(\text{val}(c_{\mathbf{u}}) + \mathbf{x} \cdot \mathbf{u}).$$

The tropicalization of $I \subseteq K[x_1, \ldots, x_n]$ is

$$\text{trop}(I) = \langle \text{trop}(f) : f \in I \rangle \subseteq \overline{\mathbb{R}}[x_1, \ldots, x_n].$$

The tropicalization of $X = V(I)$ is

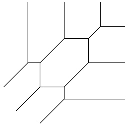
$$\text{trop}(X) = V(\text{trop}(I)) = \bigcap_{g \in \text{trop}(I)} V(g) \subseteq \overline{\mathbb{R}}^n,$$

The fundamental and structure theorems of tropical algebraic geometry

Theorem Let $X = V(I) \subseteq \mathbb{A}^n_K$. The tropicalization $\operatorname{trop}(X)$ of X equals*

$$\mathrm{cl}(\mathrm{val}(X)) = \mathrm{cl}((\mathrm{val}(x_1), \ldots, \mathrm{val}(x_n)) : x = (x_1, \ldots, x_n) \in X)$$

When X is irreducible, $\operatorname{trop}(X)$ is the support of a pure $\mathbb{R}$-rational balanced polyhedral complex of dimension $\dim(X)$ that is $\dim(X)$-connected through codimension one.



The fundamental and structure theorems of tropical algebraic geometry

Theorem Let $X = V(I) \subseteq \mathbb{A}^n_K$. The tropicalization $\mathrm{trop}(X)$ of X equals

$$\mathrm{cl}(\mathrm{val}(X(L)))$$

for any algebraically closed nontrivially valued field extension L/K.
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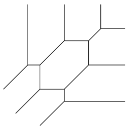
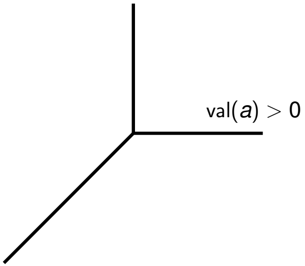


The fundamental theorem

Example $X = V(x + y - 1) \subseteq \mathbb{A}^2_{\mathbb{C}}$, where $\mathbb{C}$ has the trivial valuation.

$$X = \{(a, 1 - a) : a \in \mathbb{C}\}$$

$$\mathrm{trop}(X) = \mathrm{cl}((\mathrm{val}(a), \mathrm{val}(1 - a)) : a \in \mathbb{C}\{\{t\}\})$$



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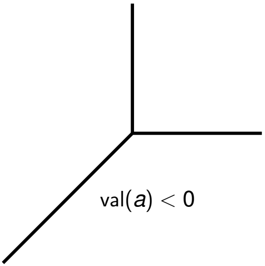

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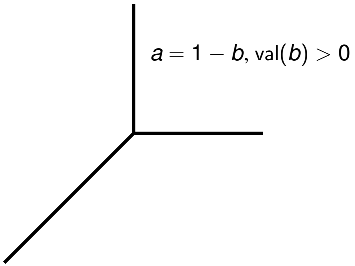
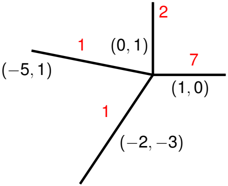


The balancing condition

A weighted one-dimensional rational polyhedral fan Σ, with s rays $\mathbb{R}_{\geq 0}\mathbf{u}_i$ weighted by m_i, is balanced if

$$\sum_{i=1}^{s} m_i \mathbf{u}_i = 0.$$

Here $\mathbf{u}_i \in \mathbb{Z}^n$ with $\gcd((\mathbf{u}_i)_j) = 1$.



$$7\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \end{pmatrix} + 1\begin{pmatrix} -2 \\ -3 \end{pmatrix} + 1\begin{pmatrix} -5 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

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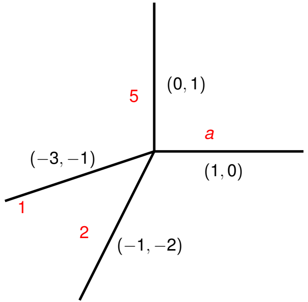
Reality check: What value of a makes this fan balanced?



Break!

Towards tropical schemes

A subscheme of $\mathbb{A}^n$ is defined by an ideal $I \subseteq K[x_1, \ldots, x_n]$:

$$X = \mathrm{Spec}(K[x_1, \ldots, x_n]/I).$$

Naive definition: A subscheme of $\mathrm{trop}(\mathbb{A}^n) = \overline{\mathbb{R}}^n$ should correspond to an ideal $I \subseteq \overline{\mathbb{R}}[x_1, \ldots, x_n]$.

Problems:

1. $\overline{\mathbb{R}}[x_1, \ldots, x_n]$ is not Noetherian
2. $\overline{\mathbb{R}}[x_1, \ldots, x_n]$ is not cancellative:
   $(x \oplus 0)^3 = (x^2 \oplus 0) \odot (x \oplus 0) = x^3 \oplus x^2 \oplus x \oplus 0.$
3. For $I \subseteq \overline{\mathbb{R}}[x_1, \ldots, x_n]$, $V(I)$ can be fairly arbitrary.

Solution: Restrict the class of ideals allowed.

Notation: $[f]_{\mathbf{x^u}}$ is the coefficient of $\mathbf{x^u}$ in f.

Definition A tropical ideal is an ideal $I \subseteq \overline{\mathbb{R}}[x_1, \ldots, x_n]$ satisfying the monomial elimination axiom:

> *for all $f, g \in I$ with $[f]_{\mathbf{x^u}} = [g]_{\mathbf{x^u}}$ there exists $h \in I$ with $[h]_{\mathbf{x^u}} = \infty$, and $[h]_{\mathbf{x^v}} \geq \min([f]_{\mathbf{x^v}}, [g]_{\mathbf{x^v}})$ (with equality if different).*

Example: If $x \oplus y, x \oplus z \in I$ then $y \oplus z \in I$.

Example: $I = \mathrm{trop}(J)$ for $J \subseteq K[x_1, \ldots, x_n]$. $f = \mathrm{trop}(F)$, $g = \mathrm{trop}(G)$, $[F]_{\mathbf{x^u}} = [G]_{\mathbf{x^u}}$. Then $h = \mathrm{trop}(F - G)$.

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Definition A subscheme of $\mathrm{trop}(\mathbb{A}^n)$ is defined by a tropical ideal in $\overline{\mathbb{R}}[x_1, \ldots, x_n]$.

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Theorem [M-Rincón]

1. Homogeneous tropical ideals have Hilbert polynomials. This leads to a definition of dimension and degree.
2. There are tropical ideals that are not $\mathrm{trop}(J)$ for any $J \subseteq K[x_1, \ldots, x_n]$.
3. If I is a tropical ideal, then $V(I)$ is the support of a finite $\mathbb{R}$-rational polyhedral complex of maximum dimension $\dim(I)$. The top-dimensional part is balanced.

Theorem (continued)

4. Tropical ideals obey the ascending chain condition: there is no
$$I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \dots$$
with all I_j tropical ideals.
5. Tropical ideals obey the weak Nullstellensatz:
$$V(I) = \emptyset \text{ if and only if } I = \langle 0 \rangle.$$
(Not true for arbitrary ideals: $V(\langle x \oplus 0, x \oplus 1 \rangle) = \emptyset$.
6. Elimination theory works for tropical ideals: $V(I \cap \overline{\mathbb{R}}[x_1, \dots, x_{n-1}]) = \pi_{n-1}(V(I))$.

Theorem (Draisma, Rincón) There are balanced $\mathbb{R}$-rational polyhedral complexes not of the form $V(I)$ for I a tropical ideal.

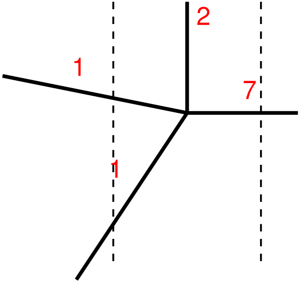
Warning: Many basic algebraic operations do not preserve the tropical ideal property.

1. If I, J are tropical ideals, then $I + J$ and $I \cap J$ might not be.
2. $(I : J)$ and $(I : J^\infty)$ might not be (problem with localization!)

Example: $I = \mathrm{trop}(\langle x - y \rangle)$, $J = \mathrm{trop}(\langle x - z \rangle)$.
$x \oplus y, x \oplus z \in I + J$, but $y \oplus z$ is not.

Balancing

A curve is balanced if $\sum m_i \mathbf{u}_i = \mathbf{0}$. Suffices to show $(\sum m_i \mathbf{u}_i) \cdot \mathbf{v} = 0$ for all $\mathbf{v} \in \mathbb{R}^n$. Equivalently: $\sum_{i:\mathbf{u}_i \cdot \mathbf{v} > 0} m_i(\mathbf{u}_i \cdot \mathbf{v}) = \sum_{i:\mathbf{u}_i \cdot \mathbf{v} < 0} m_i |\mathbf{u}_i \cdot \mathbf{v}|$.



These sums are the sums of the multiplicities of the points on the right and on the left. To show equality, it suffices to show that both sums equal the degree.

Back to the distraction ...

I zero-dimensional:

$$\deg(I) = \sum_{p \in V(I)} \mathrm{mult}_p(V(I))$$

Tropically, $\mathrm{mult}_{\mathbf{w}}(V(I)) = \deg(\mathrm{in}_{\mathbf{w}}(I))$. When $I = \mathrm{trop}(J)$, $\mathrm{mult}_{\mathbf{w}}(V(I)) = \sum_{p:\mathrm{val}(p)=\mathbf{w}} \mathrm{mult}_p(V(J))$.

Weird fact: For homogeneous $J \subseteq K[x_0, \ldots, x_n]$, $\deg((\mathrm{in}_{\mathbf{w}}(J) : x_0^{\infty})) = \sum_{p:\mathrm{val}(p) \in \mathbf{w} + \mathrm{pos}(\mathbf{e}_i : i > 0)} \mathrm{mult}_p(V(J))$.

Help!

Question: What is the right notion of prime, or equivalently of irreducibility?
Jóo-Mincheva define a prime congruence, and show that $\overline{\mathbb{R}}[x_1, \ldots, x_n]$ has Krull dimension n^*. However they also show that the varieties of primes are limited. More seriously, in forthcoming work they show that the only prime tropical ideal is the ideal of a point.
Want a definition that plays well with geometry.

Question: What about primary decomposition?