Grothendieck's localization problem

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Takeaways

① $\exists$ open problems in CA in EGA!

② Techniques from birational geometry have applications to CA!

③ $\exists$ many open problems related to this talk!

Conventions

All rings will be comm., noetherian, w/ identity

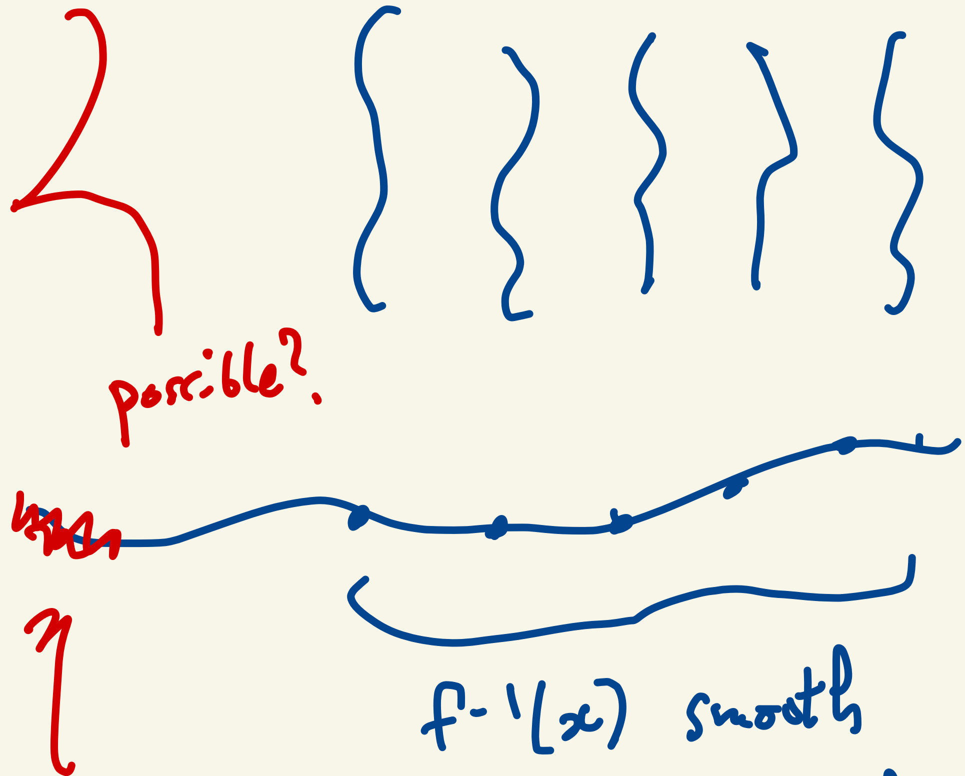
$(A, \mathfrak{m}, k)$

- k ↳ residue field $A/\mathfrak{m}$
- $\mathfrak{m}$ ↳ unique maximal ideal
- A ↳ local ring

§1 Introduction

<u>Geometric</u> Given a "nice" (i.e. proper + flat) family of alg. var.'s

$$\begin{array}{c} Y \\ f \downarrow \\ X \end{array}$$



$f^{-1}(x)$ smooth $\forall\, x \in X$ closed

<u>Q</u> Are the generic fibers also smooth?

Algebraic: Given a flat local map

$$(A, m, k) \longrightarrow (B, n, l)$$

of local rings. If

$$B \otimes_A k \text{ is regular}$$

then is $B \otimes_A \underbrace{k(p)}_{A_p / p \cdot A_p = \mathrm{Frac}(A/p)}$ regular $\forall\, p \subseteq A$ prime?

In other words, does the property of having regular fibers localize?

Two caveats

① "regular" is too weak!

generic fiber could be singular

$\exists$ (A, m, k) = local domain of dim 1

and char. $p > 0$ w/ $a \in A$ s.t.

- a has no pth root in $A/m = k$, but
- a has a pth root in Frac(A)

Then,

$$A \longrightarrow \left(\frac{A[x]}{x^p - a}\right)_{\langle m \rangle} = B$$

has regular closed fiber

$$B \otimes_A k \simeq \left(\frac{k[x]}{x^p - a}\right)$$

but non-reduced generic fiber

$$B \otimes_A \mathrm{Frac}(A) \simeq \left(\frac{\mathrm{Frac}(A)[x]}{x^p - a}\right)$$

~~Fix~~ Recall R f.t. $/k$ = field

R is smooth over k

$\iff R \otimes_k k'$ regular $\forall$ finite field extns $k \subseteq k'$

"geometrically regular"

② Version "geometrically regular" not right!

Even for our favorite local flat map

$A \to \hat{A}$

[Nagata 1962] k = field of char. $p > 0$

$$[k : k^p] = \infty$$

$$A = \left\{ \sum_{i=0}^{\infty} a_i x^i \in k[[x]] \;\middle|\; [k^p(a_0, \dots) : k^p] < \infty \right\}$$

The generic fiber

$$\hat{A} \otimes_A \mathrm{Frac}(A)$$

is not geometrically regular!

$\mathrm{Frac}(A) \subseteq \mathrm{Frac}(\hat{A})$ purely inseparable

<u>Def</u> $\mathbb{P}$ = property of local rings

k = field, R = k-alg

R is <u>geometrically $\mathbb{P}$</u> over k if local rings of $R \otimes_k k'$ are $\mathbb{P}$ $\forall$ finite field ext'ns $k \subseteq k'$.

<u>Def</u> $\varphi: R \to S$ flat is <u>geometrically $\mathbb{P}$</u> if $S \otimes_R \kappa(\mathfrak{p})$ is geom. $\mathbb{P}$ over $\kappa(\mathfrak{p})$ $\forall \mathfrak{p} \subseteq R$ prime.

Grothendieck's localization problem [1965]

$\varphi : (A, m, k) \to (B, n, l)$ flat local

2 $\left.\begin{array}{l} A \to \hat{A} \text{ geom. } \mathbb{P} \\ k \to B \otimes_A k \text{ geom. } \mathbb{P} \end{array}\right\} \Rightarrow \varphi$ geom. $\mathbb{P}$?

In other words, does the property of having geom. $\mathbb{P}$ fibers localize?

Rem Name from Avramov & Foxby [1994]

$\mathbb{P}$ = regular, normal, reduced
complete intersection, Gorenstein,
Cohen-Macaulay

what we know

P	GLP	Method
regular	André [1974]	André-Quillen homology
normal, reduced	Nishimura [1981]	— " —
complete intersections	Tabaâ [1984]	— " —
Gorenstein	Hall & Sharp [1978] + Marot [1984]	Grothendieck duality
Cohen-Macaulay	Avramov & Foxby [1994]	Cohen factorizations

Rem • [Marot 1984] follows from Hochster's conj. on ∃ small CM modules

• [Brezuleanu-Ionescu 1984; Ionescu 2008]

You can prove this using

Macaulayfications of Faltings & Kawasaki.

- A&F: finite flat dim
  cid / cmd

Q ① What about other P?
② $\exists$ uniform proof for all P above?

A YES! For well-behaved P

- $[\ell : k] < \infty$
- $A \supseteq \mathbb{Q}$

$\Big\}$ [Grothendieck & Dieudonné 1965]

$\uparrow$ [Marot 1984]

- Geometric version: φ f.t., A, B excellent [Shimomoto 2017]
- In general [M]

Next

§2 What properties $\mathbb{P}$?

§3 Techniques from birational geometry

Some proofs.

§2 What properties $\mathbb{P}$?

All properties $\mathbb{P}$ above behave well under:

- flat maps
- deformations
- localization

Permanence conditions [G&D 1965]

$\varphi : (A, m, k) \to (B, n, l)$ local flat

(I) (Ascent) $\left.\begin{array}{l} A \text{ is } \mathbb{P} \\ \varphi \text{ is geom. } \mathbb{P} \end{array}\right\} \Rightarrow B \text{ is } \mathbb{P}$

(II) (Descent) B is $\mathbb{P}$ $\Rightarrow$ A is $\mathbb{P}$

III (Deformation) A/f is $\mathbb{P}$ $\exists f$ nzd in m

$\Rightarrow A$ is $\mathbb{P}$

IV (Localization) A is $\mathbb{P} \Rightarrow A_p$ is $\mathbb{P}$

$\forall\, p \subseteq A$ prime

Handout: Known results for $\mathbb{P}$ above

- III for weak normality:
  excellent A [Bingener & Flenner 1993]
  in general [M]
- F-injective $\underbrace{\text{II} + \text{IV}}_{\substack{\text{[Hashimoto 2010]} \\ \text{CM}}}$ [Datta & M]

  [Schwede 2009] F-finite

I: [Aberbach & Enescu 2009]
Both A & φ are CM

[Datta & M] Only φ is CM

III Conj [Fedder 1983]

III is usually hardest: not satisfied

- F-pure [Fedder 1983]
- F-regular [Singh 1999]
- F-nilpotent [Srinivas & Takagi 2017]

Lots of open questions!

Main Thm [M] $\mathbb{P}$ = property of local rings

Assume
- regular $\Rightarrow \mathbb{P}$
- I ~ IV hold for $\mathbb{P}$

$\varphi : (A, \mathfrak{m}, k) \longrightarrow (B, \mathfrak{n}, l)$

$$\left.\begin{array}{l} A \to \hat{A} \text{ geom. } \mathbb{P} \\ k \to B \otimes_A k \text{ geom } \mathbb{P} \end{array}\right\} \Rightarrow \varphi \text{ geom. } \mathbb{P}$$

<u>New</u> • weakly normal

• rational sing's (e.f.t. / k of char. 0)

• seminormal

• F-rational (excellent)

• Cohen-Macaulay + F-injective

→ known previously under some finiteness conditions

§3 Techniques from birational geometry

Main strategy Reduce to A = regular.

Prop [G & D 1965]

$\varphi : (A, m, k) \to (B, n, l)$ local flat

$\mathbb{P}$ satisfies III + IV

$$\left.\begin{array}{l} A \text{ is regular} \\ B \otimes_A k \text{ is } \mathbb{P} \end{array}\right\} \Rightarrow \begin{array}{l} \text{local rings of} \\ B \otimes_A \mathrm{Frac}(A) \text{ are } \mathbb{P}. \end{array}$$

Pf $m = (x_1, x_2, \dots, x_d)$ regular sop.

$B \otimes_A k \simeq B/(x_1, \dots, x_d)B$ is $\mathbb{P}$

$\overset{\text{III}}{\Longrightarrow}$ $B/(x_1, \dots, x_{d-1})B$ is $\mathbb{P}$

$\vdots$

$\overset{\text{III}}{\Longrightarrow}$ B is $\mathbb{P}$ $\overset{\text{IV}}{\Longrightarrow}$ local rings of $B \otimes_A \mathrm{Frac}(A)$ are $\mathbb{P}$ ∎

How to reduce to this case?

Special case $A \supseteq \mathbb{Q}$

GLP is one of the first application of

Thm (Resolution of singularities [Hironaka 1964])

$(A, \mathfrak{m})$ local domain
$\cup\!|$
$\mathbb{Q}$, $A \to \hat{A}$ geom. regular

$\Rightarrow$ $\exists$ proper birational morphism
$$X \longrightarrow \operatorname{Spec}(A)$$
s.t. X is regular + integral.

$\Rightarrow$ $\exists \{A \to C_i\}_{i=1}^{m}$ finite type s.t.

① C_i regular + integral
$\operatorname{Frac}(A) = \operatorname{Frac}(C_i)$ $\forall i$

② Every pair of primes

$\mathfrak{p} \subsetneq \mathfrak{q}$ in A

lift to primes

$\tilde{\mathfrak{p}} \subsetneq \tilde{\mathfrak{q}}$ in C_i for some i.

③ ② is stable under base change.

Pf of Main Thm when $A = \mathbb{Q}$ [Marot 1984]

Special case

- $A \to \hat{A}$ has geom. regular fibers.
- A is domain and

  $B \otimes_A k(\mathfrak{p})$ is geom. $\mathbb{P}$ $\forall \mathfrak{p} \neq (0)$ prime

- Will show: local rings of

  $B \otimes_A \mathrm{Frac}(A)$ are $\mathbb{P}$.

Lem ①, ②, regular $\Rightarrow \mathbb{P}$

Then, geom. $\mathbb{P}$ maps are stable under finite type base change.

Consider

$$\begin{tikzcd}
\mathfrak{q} \arrow[r, phantom, "\subseteq"] \arrow[d, dash] & B \\
(0) \arrow[r, phantom, "\subseteq"] & A \arrow[u, "\varphi"']
\end{tikzcd}$$

$$\begin{tikzcd}
B \arrow[r] & B \otimes_A C_i \\
A \arrow[u, "\varphi"] \arrow[r] \arrow[ur, phantom, "\lrcorner", very near end] & C_i \arrow[u, "\tilde{\varphi}"']
\end{tikzcd}$$

$\tilde{\varphi}$ ← also has geom. $\mathbb{P}$ fibers away from gen. pt.

C_i regular

<u>Note</u> $\operatorname{Frac}(A) = \operatorname{Frac}(C_i)$

and so $\left\{ B_{\mathfrak{q}} = (B \otimes_A C_i)_{\tilde{\mathfrak{q}}} \right\}$

↑ STS is $\mathbb{P}$

(B)

$$\begin{tikzcd}
 & \mathfrak{q}' \arrow[rrr, dash] \arrow[dd, dash] & & & \tilde{\mathfrak{q}}' \arrow[dd, dash] \\
\mathfrak{q} \arrow[ur, phantom, "\subseteq"] \arrow[rrr, dash] \arrow[dd, dash] & & & \tilde{\mathfrak{q}} \arrow[ur, phantom, "\subseteq"] \arrow[dd, dash] & \\
 & \mathfrak{p} \arrow[rr, dash] & & \arrow[r, dash] & \mathfrak{p}' \\
(0) \arrow[ur, phantom, "\subsetneq"] \arrow[rrr, dash] & & & (0) \arrow[ur, phantom, "\subseteq"] &
\end{tikzcd}$$

Consider

$$\psi : (C_i)_{\mathfrak{p}'} \longrightarrow (B \otimes_A C_i)_{\tilde{\mathfrak{q}}'}$$

regular local ring

geom. $\mathbb{P}$ fibrs away from generic point

$$(C_i)_{\mathfrak{p}'} \supseteq \underbrace{(x_1, \ldots, x_d)}_{\text{max'l ideal}}$$

Assumption + Lemma

$\Longrightarrow (B \otimes_A C_i)_{\tilde{\mathfrak{q}}'} / (x_1, \ldots, x_d)$ is $\mathbb{P}$

III + d

$\Longrightarrow (B \otimes_A C_i)_{\tilde{\mathfrak{q}}'}$ is $\mathbb{P}$

IV

$\Longrightarrow (B \otimes_A C_i)_{\tilde{\mathfrak{q}}}$ is $\mathbb{P}$ $\blacksquare$

<u>Next case</u> A e.f.t. / field or certain DVR's

<u>Thm</u> (alterations [de Jong 1996])

$(A, \mathfrak{m})$ domain

In algebraic version of Hironaka's thm above, while we don't know whether $\{A \to C_i\}$ exists s.t.

$$\mathrm{Frac}(A) = \mathrm{Frac}(C_i),$$

we <u>do</u> know $\exists$ $\{A \to C_i\}$ s.t.

$$\mathrm{Frac}(A) \subseteq \mathrm{Frac}(C_i)$$

are finite ext'ns.

$$B_{\mathfrak{q}} \longrightarrow (B \otimes_A C_\delta)^{\sim}_{\mathfrak{q}} \quad \text{flat}$$

$\uparrow$ By ($\ddagger$) STS is $\mathbb{P}$.

Rem. · Previously used by Gabber to show Serre's non-negativity conj. [Berthelot 1997; Roberts 1998; Hochster 1997]

· Used for F-rationality in special cases [Hashimoto 2001]

General case

Thm (weak local uniformization; Gabber [Illusie, Laszlo, Orgogozo 2014])

Weak analogue of de Jong's thm in geometric setting.

BUT algebraic version we stated still holds as long as

$A \to \hat{A}$ geom. regular !

Rem Previously used to show:

$$\left.\begin{array}{l} R \quad I\text{-adically complete} \\ R/I \quad \text{quasi-excellent} \end{array}\right\} \Rightarrow R \text{ quasi-excellent}$$

(Gabber [Kurano & Shimomoto])

(R = semi-local [Rotthaus 1979]
R ⊇ $\mathbb{Q}$ [Nishimura & Nishimura 1988])

: Grothendieck's Lifting Problem"

Open Q Versions of for other properties $\mathbb{P}$.

Def Non-local ring R has **geom. $\mathbb{P}$ formal fibers** if $R_p \to \hat{R}_p$ are geom. $\mathbb{P}$ $\forall p$ prime.

Def R is **$\mathbb{P}$-2** if $\forall R \to S$ fin. type

$\{p \in \operatorname{Spec}(S) \mid S_p \text{ is } \mathbb{P}\} \subseteq \operatorname{Spec}(S)$

is open.

Conj [Imberi 1995] R I-adically complete

R/I has geom. $\mathbb{P}$ formal fibers

+ R/I is $\mathbb{P}$-2

$\Rightarrow$ Same for R

Modification of Grothendieck's Lifting Problem, based on work of Valebregu [1978]

Known R = semi-local [Brezuleanu-Ionescu 1984;
$\mathbb{P} \Rightarrow$ reduced M]

$\mathbb{P}$ = regular [Grabber]

$\mathbb{P}$ = normal [Brezuleanu & Rotthaus 1982;
Chiriacescu 1982]
+ [Nishimura & Nishimura
1988]

$\mathbb{P}$ = reduced FALSE [Nishimura 1981]