GROTHENDIECK'S LOCALIZATION PROBLEM

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Let **P** be a property of noetherian local rings.

Definition. Let k be a field, and let R be a noetherian k-algebra. We say that R is *geometrically* **P** over k if the local rings of $R \otimes_k k'$ are **P** for every finite field extension $k \subseteq k'$.

Definition. Consider a flat map $\varphi\colon R \to S$ of noetherian rings. We say that φ is *geometrically* **P** if the fiber rings $S \otimes_R \kappa(\mathfrak{p})$ are geometrically **P** over $\kappa(\mathfrak{p})$ for every prime ideal $\mathfrak{p} \subseteq R$.

Permanence conditions [EGAIV$_2$, §7]**.** Denote by $\varphi\colon (A, \mathfrak{m}, k) \to (B, \mathfrak{n}, l)$ a local flat map of noetherian local rings.

(I) (Ascent) If A is **P** and φ is geometrically **P**, then B is **P**.
(II) (Descent) If B is **P**, then A is **P**.
(III) (Deformation) If A/t is **P** for some nonzerodivisor $t \in \mathfrak{m}$, then A is **P**.
(IV) (Localization) If A is **P**, then $A_\mathfrak{p}$ is **P** for every prime ideal $\mathfrak{p} \subseteq A$.

Grothendieck's localization problem (see [EGAIV$_2$, Rem. 7.5.4(i)])**.** *Consider a flat local map $\varphi\colon (A, \mathfrak{m}, k) \to (B, \mathfrak{n}, l)$ of noetherian local rings. If both $A \to \widehat{A}$ and $k \to B \otimes_A k$ are geometrically* **P***, then is φ geometrically* **P***?*

Theorem [Mur, Thm. B]**.** *Grothendieck's localization problem holds for every property* **P** *such that regular implies* **P***, and such that* **P** *satisfies* (I)–(IV).

P	Theorem	Ascent (I)	Descent (II)	Deformation (III)	Localization (IV)
regular	[And74, Thm. on p. 297]	[Mat89, Thm. 23.7]		[EGAIV$_1$, Ch. 0, Cor. 17.1.8]	[Mat89, Thm. 19.3]
normal	[Nis81, Prop. 2.4]	[Mat89, Cor. to Thm. 23.9]		[Sey72, Prop. I.7.4]	[Bou98, Ch. V, §1, n° 5, Prop. 16]
weakly normal	[Mur, Cor. 4.12]	[Kol16, Thm. 37]	[Man80, Cor. II.2]	[Mur, Prop. 4.11]	[Man80, Cor. IV.2]
seminormal	[Mur, Cor. 4.12]	[Kol16, Thm. 37]	[GT80, Thm. 1.6]	[Hei08, Main Thm.]	[GT80, Cor. 2.2]
reduced	[Nis81, Prop. 2.4]	[Mat89, Cor. to Thm. 23.9]		[EGAIV$_2$, Prop. 3.4.6]	[Bou98, Ch. II, §2, n° 6, Prop. 17]
complete intersection	[Tab84, Thm. 2]	[Avr75, Thm. 2]		[BH98, Thm. 2.3.4(a)]	[Avr75, Cor. 1]
Gorenstein	[HS78, Thm. 3.3; Mar84, Thm. 3.2]	[BH98, Cor. 3.3.15]		[BH98, Prop. 3.1.19(b)]	[BH98, Prop. 3.1.19(a)]
Cohen–Macaulay	[AF94, Thm. 4.1]	[BH98, Thm. 2.1.7]		[BH98, Thm. 2.1.3(a)]	[BH98, Thm. 2.1.3(b)]
Cohen–Macaulay + F-injective	[Mur, Cor. 4.3]	[Ene09, Thm. 4.3]	[Has10, Lem. 4.6]	[Fed83, Thm. 3.4(1)]	[Has10, Cor. 4.11]
F-rational	[Mur, Cor. 4.5][e]	[Ene00, Thm. 2.27][e]	[DM, Prop. A.5]	[HH94, Thm. 4.2(h)][CM]	[HH94, Thm. 4.2(f)][CM]
pseudo-rational	[Mur, Cor. 4.17][char0]	[Elk78, Thm. 5][char0]	[Mur, Prop. 4.15][rc]	[Elk78, Thm. 2][char0]	[LT81, §4, Cor. of (iii)][rc]

[e] This holds for excellent rings.
[CM] This holds for homomorphic images of Cohen–Macaulay rings.
[rc] This holds for rings that have residual complexes.
[char0] This holds for rings essentially of finite type over fields of characteristic zero.

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