Geometric vertex decomposition and Liaison

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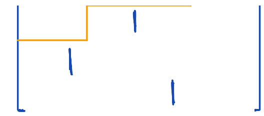
Fellowship of the Ring
Oct. 15, 2020

Based on arXiv: 2005.14289

<u>Conventions</u>: Throughout, we work over an infinite field k.

<u>I. Examples + Motivation</u>: two different perspectives/literatures on determinantal rings

Comm. Algebra	Schubert geometry
(1) $\boxed{\text{ideal of } k\times k \text{ minors of an } m\times n \text{ matrix of distinct variables}}$	$\boxed{\text{open patch of a Schubert variety in } Gr(m, m+n)}$
eg: $\left\langle 2\times 2 \text{ minors of } \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \right\rangle \subseteq k[a,b,c,d,e,f]$	eg: $\begin{bmatrix} a & b & c & 1 & 0 \\ d & e & f & 0 & 1 \end{bmatrix} \cong$ open patch of $Gr(2,5)$
(2) $\boxed{\text{One sided ladder determinantal ideals.}}$	$\boxed{\text{Schubert determinantal ideals for vexillary (2143-avoiding) permutations}}$
eg: $\begin{matrix} a & b & c & d & e & f \\ g & h & i & j & k & l \\ m & n & o & p & q & \\ r & s & t & u & v & \\ w & x & & & & \end{matrix}$ 2×2, 3×3, 2×2	eg: $w = \begin{bmatrix} 1 & & & \\ & & & 1 \\ & 1 & & \\ & & 1 & \end{bmatrix}$

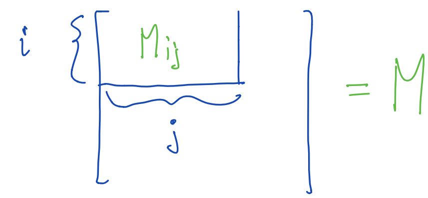


$$I = \left\langle 2\times 2 \text{ minors of } \begin{bmatrix} a & b & c & d & e & f \\ g & h & i & j & k & l \end{bmatrix} \right\rangle$$

$$+ \left\langle 3\times 3 \text{ minors of } \begin{bmatrix} a & b & c & d & e \\ g & h & i & j & k \\ m & n & o & p & q \\ r & s & t & u & v \end{bmatrix} \right\rangle$$

$$+ \left\langle 2\times 2 \text{ minors of } \begin{bmatrix} a & b \\ g & h \\ m & n \\ r & s \\ w & x \end{bmatrix} \right\rangle$$

- Given a matrix M, let M_{ij} denote the submatrix of the top i rows and left j cols.



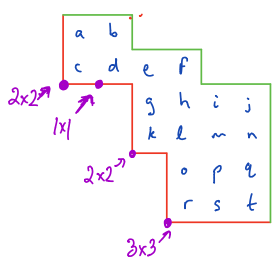
- Given a generic 7×7 matrix X of vars, define:

$$I_w = \langle 1 + \operatorname{rank}(w_{ij}) \text{ minors of } X_{ij} \rangle.$$

$$\boxed{I = I_w}$$

(3) Two-sided mixed ladder determinantal ideals.

eg:



$I = \langle$ 2x2-minors contained in $\begin{array}{cccc} a & b & & \\ c & d & e & f \end{array} \rangle$

$+ \langle$ 1x1-minors of $\begin{array}{ccc} b & & \\ d & e & f \end{array} \rangle$

$+ \langle$ 2x2-minors of $\begin{array}{cccc} e & f & & \\ g & h & i & j \\ k & l & m & n \end{array} \rangle$

$+ \langle$ 3x3-minors of $\begin{array}{ccc} f & & \\ h & i & j \\ l & m & n \\ o & p & q \\ r & s & t \end{array} \rangle$

Certain Kazhdan-Lusztig ideals

$$M_v = \left[\begin{array}{cc|c|cc|c|cc|cccc}
a & b & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
c & d & \cdot & e & f & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & g & h & \cdot & i & j & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & k & l & \cdot & m & n & \cdot & 1 & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & o & \cdot & p & q & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & r & \cdot & s & t & \cdot & \cdot & \cdot & 1 \\
\hline
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot
\end{array}\right]$$

$$\cong X_\circ^v$$

Can realize I as ideal gen. by NW minors of M_v

(to appear in an appendix to joint work in progress with Laura Escobar, Alex Fink, Alex Woo)

(4) $\boxed{\text{varieties of complexes}}$ | $\boxed{\text{certain Kazhdan-Lusztig varieties}}$

<u>Today's goal</u>: Relate two approaches used to obtain similar results about similar families of algebraic

- liaison theory
- geometric vertex decomposition.

[<u>Aside</u>: There are many interesting generalized determinantal varieties (and open qus about them) motivated by study of Schubert varieties, symmetric varieties, quiver rep varieties, degeneracy loci of v. bundles, ...

II. Geometric Vertex Decomposition

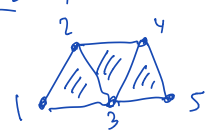
Recall the <u>Stanley-Reisner</u> correspondence

$$\text{square free mon. ideal } I_\Delta \subseteq \underbrace{k[x_1,\dots,x_n]}_{R} \longleftrightarrow \text{simplicial complex } \Delta \text{ on vertex set } \{1,2,\dots,n\}$$

<u>correspondence by</u>:

$$x_{i_1}x_{i_2}\cdots x_{i_r}\in I_\Delta \iff \{i_1,i_2,\dots,i_r\}\notin\Delta$$

<u>eg</u>: $I_\Delta = \langle x_1x_4,\ x_2x_5,\ x_1x_5\rangle \subseteq k[x_1,\dots,x_5]$

$\Delta =$ 

<u>Key idea</u>: Good combinatorial properties of Δ yield good commutative alg. properties of R/I_Δ.

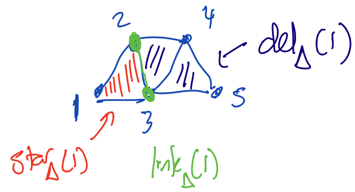
<u>Today</u>: Focus on <u>vertex decomposability</u> of Δ

<u>Thm</u>: If Δ is vertex decomp, then R/I_Δ is Cohen-Macaulay.

<u>Def</u>: Let Δ be a simplicial complex, $v\in\Delta$ be a vertex

- $\mathrm{star}_\Delta(v) = \{F\in\Delta \mid F\cup\{v\}\in\Delta\}$
- $\mathrm{del}_\Delta(v) = \{F\in\Delta \mid F\cap\{v\}=\emptyset\}$
- $\mathrm{link}_\Delta(v) = \{F\in\Delta \mid F\cup\{v\}\in\Delta,\ F\cap\{v\}=\emptyset\}$

eg: $v=1$, Δ as above



"vertex decomposition at 1"

"is vertex decomposable."

$$I_\Delta = \langle x_4, x_5\rangle \cap \langle x_1, x_2x_5\rangle = I_{star_\Delta(1)} \cap I_{del_\Delta(1)}.$$

Def: A pure simplicial complex is <u>vertex decomposable</u> if

(i) $\Delta = \emptyset$ or Δ is a simplex OR

(ii) $\exists v \in \Delta$ s.t. $lk_\Delta(v)$ and $del_\Delta(v)$ are vertex decomposable.

Next: an analog of vertex decomposability for more general varieties.

Def: (Knutson - Miller - Yong '05)

- Fix the lex order $x_1 > x_2 > x_3 > \cdots > x_n$ on $k[x_1, x_2, \dots, x_n] = R$
- Let $I \subseteq R$ be an ideal.
- Let $G = \{ x_1^{d_i} q_i + r_i \mid 1 \le i \le m\}$ be a GB for I, where $x_1 \nmid q_i$ and $in_{x_1}(x_1^{d_i}q_i + r_i) = x_1^{d_i} q_i$
- Let $C_{x_1,I} = \langle q_i \mid 1 \le i \le m\rangle$, $N_{x_1,I} = \langle q_i \mid d_i = 0\rangle$

When $in_{x_1} I = C_{x_1,I} \cap (N_{x_1,I} + \langle x_1\rangle)$, this decomposition is called a <u>geometric vertex decomposition</u>.

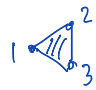
Observe: When $I = I_\Delta$ is a Stanley-Reisner ideal,

$$C_{x_1,I_\Delta} = I_{star_\Delta(1)}, \quad N_{x_1,I_\Delta} + \langle x_1\rangle = I_{del_\Delta(1)}.$$

eg: $I = \left\langle 2\times 2 \text{ minors of } \begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{bmatrix}\right\rangle \xrightarrow{lex_<} I_\Delta$ for $\Delta = \mathrm{cone}_4($  $)$ "Δ'"

$$in_{x_1} I = \langle x_1x_5, x_1x_6, x_2x_6 - x_3x_5\rangle$$
$$= \underbrace{\langle x_5, x_6\rangle}_{<} \cap \underbrace{\langle x_1, x_2x_6 - x_3x_5\rangle}_{<} \quad \leftarrow \text{geometric vertex decomp.}$$

$\langle x_5, x_6 \rangle$ $\qquad\qquad$ $\langle x_1, x_2 x_6 \rangle$



$\text{star}_{\Delta'}(1)$ $\qquad\qquad$ $\text{del}_{\Delta'}(1)$ $\qquad$ on vertex set $\{1,2,3,5,6\}$.

<u>Idea</u>: geometric vertex decomp. provides a geometric explanation for a vertex decomp of the simplicial complex $\text{in}_< I$.

<u>Def</u>: An unmixed ideal $I \subseteq R = k[x_1, \dots, x_n]$ is <u>geometrically vertex decomposable</u> if

(i) $I = (1)$ or I is gen by indeterminates OR

(ii) for some $y = x_i$, $\exists$ lex order $y > x_{i_2} > x_{i_3} > \cdots > x_{i_n}$ s.t. y divides some term in the reduced GB for I and $\text{in}_y I = C_{y,I} \cap (N_{y,I} + \langle y \rangle)$ is a geom. vertex decomp and C and N are geometrically vertex decomp.

<u>Examples</u>: The following are geom vertex decomp.

- Stanley-Reisner ideals of vertex decomp. complexes
- determinantal ideals
- ladder det. ideals
- Schubert det ideals
- Kazhdan - Lusztig ideals
- ideals of lower bound cluster algs.
- type A quiver ideals
- any ideal I s.t. $\text{in}_< I$ is the Stanley-Reisner ideal of a vertex decomposable simplicial complex with a "compatible" order on vertices.

<u>Different sort of eg</u>: $I = \langle y(zs - x^2), ywr, wr(zx + s^2 + z^2 + wr) \rangle$

- there are no squarefree initial ideals
- I <u>is</u> geometrically vertex decomp.

Prop: Geometrically vertex decomp. ideals are radical.

Later: Homog. geometrically vertex decomp. ideals are Cohen-Macaulay (in fact, they are glicci)

III Gorenstein Liaison (very briefly)

Def: Let $V_1, V_2, X \subseteq \mathbb{P}^n$ be subschemes def. by sat, homog. ideals $I_{V_1}, I_{V_2}, I_X \subseteq R$, and assume X is arithmetically Gorenstein. If $I_X \subseteq I_{V_1} \cap I_{V_2}$ and $[I_X : I_{V_1}] = I_{V_2}$, $[I_X : I_{V_2}] = I_{V_1}$ then V_1, V_2 are directly algebraically G-linked by X.

Def: If there is a sequence of G-links from I_{V_1} to a complete intersection, then say that I_{V_1} is glicci.

Thm: I glicci $\Rightarrow$ I is Cohen-Macaulay

($\overset{?}{\Leftarrow}$ open)

Def: Let I, C be homog, saturated, unmixed ideals of R with $ht(I) = ht(C)$.

Suppose $\exists$ homog. CM ideal $N \subseteq I \cap C$ of $ht(I) - 1$ and an isom. $I/N \cong C/N(-1)$ as graded R/N-modules.

If N is G_0, say that I is obtained from C by an elementary G-biliaison of height 1.

Thm (Hartshorne) For I, C as above, I is G-linked to C in 2 steps.

eg: $I = \langle 2\times 2$ minors of $\begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{bmatrix}$

$C = C_{x_1, I} = \langle x_5, x_6 \rangle$, $N = N_{x_1, I} = \langle x_2x_6 - x_3x_5 \rangle$

Then
- $N \subseteq I \cap C$
- $ht\, C = ht\, I = 2$, $ht\, N = 1$

• $\varphi : C \to I/N$, $f \mapsto f\left(\frac{x_1x_5 - x_2x_4}{x_5}\right)$.

Check: this is surjective with kernel N.

IV Geometric vertex decomposition & liaison

Nagel–Römer: Stanley–Reisner ideals of (weakly) vertex decomposable complexes are glicci.

Gorla, Migliore, Nagel: • many generalized det. ideals are glicci
• used liaison to obtain GB of classes of generalized det. ideals.

Thm (Klein–R). • Under mild hypotheses, every geometric vertex decomp. gives rise to an elementary G-biliaison of height 1.
• Every sufficiently nice elementary G-biliaison of ht 1 gives rise to a geometric vertex decomp.

Cor: Homogeneous (weakly) geometrically vertex decomposable ideals are glicci ($\therefore$ CM)

Cor:

Let $I = \langle yq_1 + r_1, \ldots, yq_k + r_k, h_1, \ldots, h_e \rangle$ be a homog ideal in $R = k[x_1, \ldots, x_n]$ and $y = x_i$, and assume that y doesn't divide any q_i, r_i, h_i.

Fix a lex order $y > x_{i_2} > \cdots > x_{i_n}$ and suppose

$$G_C = \{q_1, \ldots, q_k, h_1, \ldots, h_e\}, \quad G_N = \{h_1, \ldots, h_e\}$$

are Gröbner bases for the ideals they gen, which we call C, N.
Assume $ht(I), ht(C) > ht(N)$ and that N has no embedded primes. Let $M = \begin{pmatrix} q_1 & \cdots & q_k \\ r_1 & \cdots & r_k \end{pmatrix}$. If the ideal of 2-minors of M is contained in N, then the given gens of I are a GB.

See Patricia Klein's recent preprint for an application where this is used. (arXiv: 2008.01717)