Minimal exponents and a conjecture of Teissier

(jt. work with Eva Elduque and Brad Dirks)

I. Setup

$0 \neq f \in R = \mathbb{C}[x_1, \ldots x_n]$, $f(0) = 0$

$0 = P \in H = (f = 0)$

Interested in invariants that measure the sing of H at P

<u>Example</u> multiplicity $\mathrm{mult}_P(f) = \max\{r \mid f \in (x_1, \ldots x_n)^r\}$

Note: $P \in H$ smooth iff $\mathrm{mult}_P(f) = 1$

II. The Bernstein – Sato polynomial

Consider $B_f = R_f[s] \cdot f^s$ free module / $R_f[s]$ basis f^s

The Weyl algebra $D_R = \mathbb{C}\langle x_1, \dots x_n, \partial_{x_1}, \dots \partial_{x_n} \rangle$ acts on B_f via:

$$\partial_{x_i} \cdot f^s = \text{"} s \frac{\partial f}{\partial x_i} f^{s-1} \text{"} = \frac{s \frac{\partial f}{\partial x_i}}{f} f^s$$

key result in D-module thy:

$\exists$ $b(s) \neq 0$ such that $b(s) f^s \in D_R[s] \cdot f^{s+1}$

The monic gen. of the ideal of such $b(s)$ is the Bernstein – Sato poly $b_f(s)$.

<u>Examples</u> 1) $f = x_1$ (more gen: smooth hyp)

$$\partial_{x_1} \cdot f^{s+1} = (s+1) f^s \Rightarrow b_f(s) \mid (s+1)$$

In fact, $b_f(s) = (s+1)$ because of

<u>Rmk</u> $(s+1) \mid b_f(s)$: $b_f(s) f^s \in D_R[s] \cdot f^{s+1}$

Specialize to $s = -1$: $b_f(-1) \dfrac{1}{f} \in D_R \cdot 1 = R$

$$\Rightarrow b_f(-1) = 0$$

2) $f = x_1^2 + \dots + x_n^2$

$$\partial_{x_i} \cdot f^{s+1} = (s+1) 2 x_i f^s \Rightarrow \partial_{x_i}^2 f^{s+1} = 2(s+1) f^s + s(s+1) 4 x_i^2 f^{s-1}$$

$$\Rightarrow \left(\sum \partial_{x_i}^2\right) f^{s+1} = (s+1)(2n + 4s)$$

$$b_f(s) \mid (s+1)\left(s + \frac{n}{2}\right)$$

We have equality :

<u>Fact:</u> $b_f(s) = (s+1)$ iff H smooth

3) $f = \det(x_{ij}) \in \mathbb{C}[x_{ij} \mid 1 \leq i,j \leq n]$

$$b_f(s) = (s+1)(s+2)\cdots(s+n)$$

(Cayley) $b_f(s) f^s = \det(\partial_{i,j}) \cdot f^{s+1}$

III. The minimal exponent

<u>Thm</u> (Kashiwara) All roots of $b_f(s) \in \mathbb{Q}_{<0}$

One can be more precise:

suppose we have an embedded res. of sing

$$\begin{array}{l} Y \text{ smooth} \\ \pi \big\downarrow \\ \mathbb{C}^n \end{array}$$

π birat, proper such that locally on Y have coord $y_1, \dots, y_n$ s.t

$$f \circ \pi = \text{invert} \cdot \prod y_i^{a_i}$$

$$\det(\mathrm{Jac}_\pi) = \text{invert} \cdot \prod y_i^{k_i}$$

Lichtin: every root of $b_f(s)$ is of the form

$$-\frac{k_i + 1 + m}{a_i} \quad \text{for some } i, \; m \in \mathbb{Z}_{\geq 0}$$

In particular, it is $\leq -\underbrace{\min_i \frac{k_i+1}{a_i}}_{\text{log canonical threshold } \mathrm{lct}(f)}$

Recall: $\mathrm{lct}(f)$ is indep of resol.

In fact it is given by

$$\mathrm{lct}(f) = \sup\left\{ s>0 \;\middle|\; \frac{1}{|f(x)|^{2s}} \text{ locally integr} \right\}$$

Kollár: used this descr. of lct + integr by parts to show $b_f(-\mathrm{lct}(f)) = 0$

Conclusion: the largest root of $b_f(s)$ is $-\mathrm{lct}(f)$

<u>Def</u> (Saito) The minimal exponent of f, den. $\tilde{\alpha}(f)$, is the negative of the largest root of $b_f(s)/(s+1)$.

Convention: if $b_f(s) = (s+1)$ ($\Leftrightarrow$ H smooth) then $\tilde{\alpha}(f) = \infty$

Note: $\quad \mathrm{lct}(f) = \min\{1, \tilde{\alpha}(f)\}$

The log canonical threshold measures how far the pair (X, H) is from being log canonical.

The min exp is interesting when $\mathrm{lct}(f) = 1$, when it gives more refined info.

Thm (Saito) $\quad \tilde{\alpha}(f) > 1$ iff H has rational sing.

Examples $\quad$ 1) $\quad f = x_1^{a_1} + \dots + x_n^{a_n}, \quad a_i \geq 2 \ \forall i$

$$\Rightarrow \tilde{\alpha}(f) = \sum_{i=1}^{n} \frac{1}{a_i}$$

2) f homog, deg $d \geq 2$, w. isol. sing at 0

$$\Rightarrow \tilde{\alpha}(f) = \frac{n}{d}$$

Rmk. When f has isol sing, $\tilde{\alpha}(f)$ is known as Arnold exponent, studied in 80s by Varchenko, Steenbrink, Loeser. It is the first spectral number.

Rmk. In jt. work with M. Popa we extended properties of lct to min exponent (e.g. semicontinuity) using Hodge ideals

Open questions about min exponents:

1) Description of $\tilde{\alpha}(f)$ via valuations
2) Analyt. descr. of $\tilde{\alpha}(f)$ (for isol. sing : Loeser)
3) Description via Igusa/motivic type zeta functions.
4) Analogue in char p?

IV. Teissier's conjecture

From now on: suppose $0 \in H$ isol. sing.

After replacing $\mathbb{C}^n$ by a small nghd of 0,

get local versions of the invar: $\tilde{\alpha}_p(f)$, $lct_p(f)$.

Teissier's invariant $\theta_p(f)$:

$$J_f = \left(\frac{\partial f}{\partial x_1}, \dots \frac{\partial f}{\partial x_n} \right), \quad (x_1, \dots x_n) \subseteq \sqrt{J_f}$$

integral closure

$$\theta_0(f) = \min \left\{ \frac{r}{s} \;\middle|\; \underset{\text{around } 0}{(x_1, \dots x_n)^r \subseteq \overline{J_f^s}} \right\}$$

$$= \max \left\{ \frac{ord_E(J_f)}{ord_E(x_1, .. x_n)} \;\middle|\; \begin{array}{c} E \text{ divisor over } \mathbb{C}^n \\ \text{w. center at } 0 \end{array} \right\}$$

Note: if $P \in H$ smooth $\Rightarrow \theta_p(f) = 0$

if $\mathrm{mult}_p(f) = d \geq 2 \Rightarrow J_f \subseteq (x_1, \dots x_n)^{d-1} \Rightarrow \theta_p(f) \geq d-1$

<u>Conjecture</u> (Teissier, '80) For every hyperplane $0 \in Y \subset \mathbb{C}^n$

$$\tilde{\alpha}_p(f) \geq \tilde{\alpha}_p(f|_Y) + \frac{1}{\theta_p(f)+1}$$

In particular: if $Z_1, \dots Z_{n-1}$ general hyp through 0

$$\Rightarrow \tilde{\alpha}_p(f) \geq \frac{1}{\theta_p(f)+1} + \frac{1}{\theta_p(f|_{Z_1})+1} + \dots + \frac{1}{\theta_p(f|_{Z_1 \cap \dots \cap Z_{n-1}})+1}$$

Example $f = x_1^{a_1} + \dots + x_n^{a_n}$ $a_1 \geq \dots \geq a_n \geq 2$

$Y = (x_1 = 0) \Rightarrow \tilde{\alpha}_P(f|_Y) = \frac{1}{a_2} + \dots + \frac{1}{a_n}$

$J_f = (x_1^{a_1 - 1}, \dots, x_n^{a_n - 1})$, $\theta_P(f) = a_1 - 1$

$\Rightarrow$ have equality in Teissier's conj.

With E. Elduque: version for lcts

B. Dirks : full conjecture.

Loeser proved a weaker version:

$$\tilde{\alpha}_p(f) \geq \tilde{\alpha}_p(f|_Y) + \frac{1}{\lceil \theta_p(f) \rceil + 1}$$

Sketch of Loeser's proof: assume $Y = (x_n = 0)$

consider the family of polyn

$$h_t(x_1, \dots, x_n) = f(x_1, \dots x_{n-1}, t x_n) + (1-t) x_n^m, \quad m = \lceil \theta_p(f) \rceil + 1$$

$$h_0 = f|_Y + x_n^m$$

Thom - Sebastiani: $\tilde{\alpha}_p(h_0) = \tilde{\alpha}_p(f|_Y) + \frac{1}{m}$

$$h_1(x_1, \dots x_n) = f$$

Key pt: hyp on $m \Rightarrow$ for t in a nghd of 1
the Milnor # $\mu(h_t) = \dim(R/J_{h_t})_0$ is constant

Result of Varchenko, Steenbrink: in this case we have
$\widetilde{\alpha}_p(h_t)$ is const $= \widetilde{\alpha}_p(h_1)$ in a nghd of 1

By semicont of $\widetilde{\alpha}_p \Rightarrow$

$$\begin{array}{ccc} \widetilde{\alpha}_p(h_1) & \geq & \widetilde{\alpha}_p(h_0) \\ \| & & \| \\ \widetilde{\alpha}_p(f) & & \widetilde{\alpha}_p(f|_{H}) + \frac{1}{m} \end{array}$$

Pf. of the key pt: $\quad h_t = f(x_1, \dots x_{n-1}, t x_n) + (1-t) x_n^m$

$t \neq 1$, h_t has isol sing $\quad J_{h_t}$: $\dfrac{\partial f}{\partial x_i}(x_1, \dots x_{n-1}, t x_n) \quad i \leq n-1$

$$\frac{\partial f}{\partial x_n}(x_1, \dots x_{n-1}, t x_n)\, t + m(1-t) x_n^{m-1}$$

change var: $x_n \longrightarrow \frac{1}{t} x_n \implies$

$$J_{h_t} \rightsquigarrow \left(\frac{\partial f}{\partial x_1}, \dots \frac{\partial f}{\partial x_{n-1}}, \frac{\partial f}{\partial x_n} + ct\, x_n^{m-1} \right) \subseteq \overline{J_f}$$

$\implies \underset{\mu_p(f)}{\overset{\shortparallel}{e(J_f)}} \leq \underset{\mu_p(h_t)}{\overset{\shortparallel}{e(J_{h_t})}}$; $\mu_p(f) \geq \mu_p(h_t)$ in a nghd of $t=1$ by semicont. $\implies$ QED

Idea of pf. of

$$\operatorname{lct}_p(f) \geq \min\left\{1,\ \operatorname{lct}_p(f|_Y) + \frac{1}{\theta_p(f)+1}\right\}$$

Choose $d \geq 1$ such that $d \cdot \theta_p(f) \in \mathbb{Z}$ and consider the family

$$h_t(x_1, \dots x_n) = f(x_1, \dots x_{n-1}, t x_n^d) + (1-t) x_n^m, \text{ where}$$

$$m = d(\theta_p(f) + 1)$$

$$h_1 = f(x_1, \dots x_{n-1}, x_n^d)$$

Here $\mathcal{J}(f^\lambda)$ are the multiplier ideals of f

$$\operatorname{lct}_p(f) > c \iff \mathcal{J}(f^c) = R \text{ at } 0$$

$$\Updownarrow$$

$$x_n^{d-1} \in \mathcal{J}(h_1^c) \text{ at } 0$$

$$h_t(x_1, \dots x_n) = f(x_1, \dots x_{n-1}, t x_n^d) + (1-t) x_n^m \qquad Y = x_n = 0$$

$$h_0 = f|_Y + x_n^m$$

Thom - Sebastiani for mult. ideals $\Rightarrow x_n^{d-1} \in \mathcal{J}(h_0^c)$

if $c < \min \left\{ 1, \, lct_0(f|_Y) + \dfrac{1}{\theta_0(f)+1} \right\}$

Using Restr. thm for mult. ideals

+ work $\Rightarrow x_n^{d-1} \in \mathcal{J}(h_t^c)$ for t in a nghd of 0

Again: hyp on $m \Rightarrow \mu(h_t)$ is const in a nghd of $t = 1$

$\Rightarrow \mathcal{J}(h_t^c)$ vary in a flat family in nghd of 1

$\Rightarrow x_n^{d-1} \in \mathcal{J}(h_1^c) \Rightarrow lct_p(f) > c \qquad \square$

To get the full statement:

proceed similarly via Hodge ideals instead of mult. ideals.