A toric BGG correspondence

(Joint with D. Eisenbud, D. Erman, F.O. Schreyer)

k a field

$S = k[x_0, \dots, x_n]$, $|x_i| = 1$

$E = \Lambda_k(e_0, \dots, e_n)$, $|e_i| = -1$

Thm (Bernstein-Gel'fand-Gel'fand, 78) $D^b(S) \simeq D^b(E)$.

- Eisenbud-Fløystad-Schreyer, 03: use BGG to give an efficient algorithm for computing sheaf cohomology on $\mathbb{P}^n$
- Green, 98: uses BGG to bound the length of the linear strand of min'l free resolutions over S (Linear Syzygy Theorem)

Goals:

- Extend the EFS algorithm from $\mathbb{P}^n$ to weighted proj. space.
- Use a non-standard gdd version of BGG to bound the length of virtual resolutions over coordinate rings of toric varieties, answering a question of Berkesch-Erman-Smith.

Background on BGG

Com(S): complexes of gdd S-modules

Com(E): " " E-modules

$$L : \mathrm{Com}(E) \rightleftarrows \mathrm{Com}(S) : R$$

$N \in \mathrm{Mod}(E)$. $L(N)_i = N_i \otimes_k S(-i)$, $\partial_L = \sum_{i=0}^{n} \cdot e_i \otimes x_i$.

$M \in \mathrm{Mod}(S)$, $R(M)_i = M_{-i} \otimes_k \omega(i)$, $\partial_R = \sum_{i=0}^{n} \cdot x_i \otimes e_i$

$\omega = \mathrm{Hom}_k(E, k) \cong E(-n-1)$

Ex: $n=1$, $N = E$.

Basis of E: $\{1, e_0, e_1, e_0e_1\}$
$\quad 0 \quad -1 \quad -1 \quad -2$

$$L(N) = 0 \to \underset{0}{1 \otimes S} \xrightarrow{\begin{pmatrix} x_0 \\ x_1 \end{pmatrix}} \begin{matrix} e_0 \otimes S(1) \\ \oplus \\ e_1 \otimes S(1) \end{matrix} \xrightarrow{(-x_1 \;\; x_0)} e_0e_1 \otimes S(2) \to 0$$

(dual) Koszul complex

$n=1$, $M = S$

$$R(M) = 0 \to \underset{0}{1 \otimes \omega} \xrightarrow{\begin{pmatrix} e_0 \\ e_1 \end{pmatrix}} \begin{matrix} x_0 \otimes \omega(-1) \\ \oplus \\ x_1 \otimes \omega(-1) \end{matrix} \to \cdots$$

If $C \in \mathrm{Com}(S)$, $C' \in \mathrm{Com}(E)$

$LR(C)$ is a free res'n of C

$RL(C')$ is an inj. res'n of C'.

$\rightsquigarrow$ L, R induce equiv. $D^b(S) \simeq D^b(E)$.

Special case of Koszul duality: if R is a Koszul alg/k,

$$D^b(R) \simeq D^b(\mathrm{Ext}_R^*(k,k)).$$

BGG for nonstandard gradings

A abelian group

$S = k[x_0, \dots, x_n]$, A-graded

Ex: $n=1$, $A = \mathbb{Z}$, $|x_0| = 1$, $|x_1| = 2$.

Naive attempt at defining $R : \mathrm{Com}(S) \longrightarrow \mathrm{Com}(E)$:

$E = \Lambda_k(e_0, e_1)$, $|e_0| = -1$, $|e_1| = -2$.

$R(M)_i = M_{-i} \otimes_k \omega(i)$, $\quad \partial_R = x_0 \otimes e_0 + x_1 \otimes e_1$.

$$\begin{tikzcd}
M_0 \otimes \omega \arrow[r, bend right, "x_0 \otimes e_0"'] \arrow[rr, bend left, "x_1 \otimes e_1"] & M_1 \otimes \omega(-1) & M_2 \otimes \omega(-2) \\
0 & -1 & -2
\end{tikzcd}$$

∂_R does not respect the homological grading!

Our solution: <u>forget</u> the homological grading.

$E = \Lambda_k(e_0, \dots, e_n)$, $A \times \mathbb{Z}$-gdd, $|e_i| = (-|x_i|, 1)$

Def'n: A <u>differential</u> <u>E-module</u> is a pair (D, ∂), where D is an $A \times \mathbb{Z}$-gdd E-module, and $\partial : E \longrightarrow E(0,1)$ s.t. $\partial^2 = 0$.

$DM(E)$: cat. of differential E-modules.

$$L : DM(E) \rightleftarrows \mathrm{Com}(S) : R$$

$$L(D, \partial_D)_i = \bigoplus_{a \in A} D_{(a,i)} \otimes_k S(-i), \quad \partial_L = \sum_{i=0}^{n} e_i \otimes x_i + (-1)^i \partial_D.$$

$M \in \mathrm{Mod}(S)$

$$R(M) = \bigoplus_{a \in A} M_{-a} \otimes_k \omega(a,0) \in DM(E)$$

$$\partial_R = \sum_{i=0}^{n} x_i \otimes e_i .$$

$DM(E) \simeq$ cat. of cpx's of the form

$$\cdots \xrightarrow{\partial} M(0,-1) \xrightarrow{-\partial} M \xrightarrow{\partial} M(0,1) \xrightarrow{-\partial} \cdots ,$$

See also Baranovsky, 07.

Prop: $D^b(S) \simeq D^b_{DM}(E)$

when S is st. gdd, $D^b_{DM}(E) \simeq D^b(E)$

Geometric applications

$S = k[x_0, \ldots, x_n]$, st. gdd

$$\begin{tikzcd} D^b(S) \arrow[r, phantom, "\simeq"] \arrow[d] & D^b(E) \arrow[d] \arrow[rr, bend left, squiggly, "C"] & & K^{ex}(E) \\ D^b(\mathbb{P}^n) \arrow[r, phantom, "\simeq"] & \underbrace{D^b(E)/\mathrm{Perf}(E)}_{\text{singularity cat. of } E.} \arrow[rr, "\simeq"] & & \underbrace{K^{ex}(E)}_{\text{exact cpx's of f.g. free } E\text{-mod's} / \text{htpy}} \end{tikzcd}$$

Buchweitz, 86

$$D^b(\mathbb{P}^n) \xrightarrow{\simeq} K^{ex}(E)$$

$\mathcal{F} \in \mathrm{coh}(\mathbb{P}^n) \rightsquigarrow$ corresponding minimal cpx in $K^{ex}(E)$, denoted $T(\mathcal{F})$, called the Tate res'n of $\mathcal{F}$ (EFS, 03).

$C : D^b(E) \longrightarrow K^{ex}(E)$ is defined as follows:

$N \in D^b(E)$.

$$\left.\begin{array}{l} F \xrightarrow{\simeq} N \\ G \xrightarrow{\simeq} N^\vee = \mathcal{H}om_E(N, E) \end{array}\right\} \text{min'l free res'ns of } N$$

$$C(N) = \mathrm{Cone}\left(F \xrightarrow{\simeq} N \cong N^{\vee\vee} \xrightarrow{\simeq} G^\vee\right) \in K^{ex}(E).$$

If $\mathcal{F} \in \mathrm{coh}(\mathbb{P}^n)$, and $\mathcal{F} = \widetilde{M}$, $M \in \mathrm{mod}(S)$

$$T(\mathcal{F}) = C(R(M))$$

Thm (EFS, 03) $\dim_k H^i(\mathbb{P}^n, \mathcal{F}(j)) =$ # of copies of $\omega(-j)$ in $T(\mathcal{F})_{-i-j}$.

X : smooth proj. toric variety w/ irrelevant ideal η

$S =$ Cox ring of X

$S = k[x_0, \dots, x_n]$, $A = \mathrm{Pic}\, X$ gdd

$$\begin{array}{ccc} D^b(S) & \simeq & D^b_{DM}(E) \\ \downarrow & & \downarrow \\ D^b(X) & \simeq & D^b_{DM}(E) / R(\text{cpx's w/ irrelevant homology}) \simeq\ ? \end{array}$$

No obvious analogue of Buchweitz equiv.

When $X =$ weighted proj. space

(i.e. $A = \mathbb{Z}$, $S = k[x_0, \dots, x_n]$, $|x_i| \geq 1$)

$\eta = (x_0, \dots, x_n)$, and we can take

$? = K^{ex}_{DM}(E)$, cat. of exact, free, locally fg DM's / htpy.

[Added note: weighted proj. space is not smooth! But smoothness isn't really necessary here. Just replace Pic X w/ Cl(X)]

Def'n: (Avramov - Buchweitz - Iyengar, 07) A <u>free flag</u> is an object $(F,\partial) \in DM(E)$ where F is free and equipped w/ a decomp. $F = \bigoplus_{i \geq 0} F_i$ s.t.

$$\partial(F_i) \subseteq \bigoplus_{j<i} F_j.$$

Ex: If $\partial(F_i) \subseteq F_{i-1}$, then (F,∂) is just a right bounded complex.

If $(D,\partial_D) \in DM(E)$, a free flag res'n is a quasi-isom. $(F,\partial_F) \xrightarrow{\simeq} (D,\partial_D)$, where (F,∂_F) is a free flag.

Free flag res'ns always exist, but <u>minimal</u> free flag res'ns do not!

Def'n: A min'l free res'n of $(D,\partial_D) \in DM(E)$ is a quasi-isom. $(G,\partial_G) \xrightarrow{\simeq} (D,\partial_D)$, where

- ∂_G is minimal, G is free, and
- $\exists$ free flag res'n $(F,\partial_F) \xrightarrow{\simeq} (D,\partial_D)$ s.t. $(F,\partial_F) \cong (G,\partial_G) \oplus (C,\partial_C)$, (C,∂_C) contractible.

Thm (BEES): Min'l free res'ns of DM's exist and are unique up to isom.

Back to Tate res'ns over weighted proj. sp.

If $X = \mathbb{P}(w_0, \dots, w_n)$, and $\mathcal{F} \in \mathrm{coh}(X)$, choose $M \in \mathrm{mod}(S)$ s.t. $\widetilde{M} = \mathcal{F}$. Choose a min'l res'n

$$F \xrightarrow{\simeq} R(M)$$

$$T(\mathcal{F}) = \mathrm{cone}(F \xrightarrow{\simeq} R(M)) \in K^{ex}_{DM}(E).$$

Thm (BEES) $\dim_k H^i(X, \mathcal{F}(j)) = \#$ of copies of $\omega(-j, -i)$ in $T(\mathcal{F})$.

Virtual res'ns

X : smooth proj. toric variety w/ irrel. ideal B.

S : Cox ring of X

Def'n (Berkesch-Erman-Smith) A virtual res'n of an S-mod. M is a free cpx $F \to M$ s.t. $\widetilde{F} \to \widetilde{M}$ is a loc. free res'n.

Question (BES): If $M \in \mathrm{mod}(S)$, does M have a virtual res'n of length $\leq \dim X$?

BES: "yes" for products of proj. spaces

Thm (BEES) "Yes".

Sketch of pf: $H(R(M))_{(a,\underline{j})} = \mathrm{Tor}^S_{-j}(M,k)_a$

$\Rightarrow$ length of min'l res'n of M is determined by the "$\mathbb{Z}$-degrees" appearing in $H(R(M)) \in \mathrm{Mod}(E)$.

Can show: for $d \gg 0$, $H(R(M_{\geq d}))$ ($d \in \mathrm{Pic}\,X$) is a subquotient of $\mathfrak{n}_E R(M)$, where $\mathfrak{n}_E$ = "exterior irrelevant ideal".

$\mathfrak{n}_E$ gen. by degree r monomials, $r = \mathrm{rk}\,\mathrm{Pic}\,X$.

$\Rightarrow$ $H(R(M))$ lives in at most $n-r+1$ "$\mathbb{Z}$-degrees".

$\Rightarrow$ $M_{\geq d}$ has a min'l res'n of length $n-r = \dim X$. $\square$