Strata Separation for W.P.

joint with K. Bromberg.

Let $S = S_{g,n}$ surface of finite type

$\mathrm{Teich}(S)$ = space of marked conformal str on S.

Weil Petersson metric.

Neg curved, Incomplete

(Wolpert, Takhtajan)

Completion $\overline{\mathrm{Teich}(S)}$ is Deligne-Mumford cptification. (Masur)

Have α closed curve

$$\ell_\alpha : \mathrm{Teich}(S) \to \mathbb{R}_+$$

extends to

$$\ell_\alpha : \overline{\mathrm{Teich}(S)} \to [0,\infty]$$

If τ multicurve,

$$\mathcal{S}_\tau = \left\{ X \in \overline{\mathrm{Teich}(S)} \;\; \ell_\alpha(X) = 0 \iff \alpha \subseteq \tau \right\}$$

$\mathcal{S}_\tau \subseteq \overline{\mathrm{Teich}(S)}$ is stratum of noded Riemann surfaces

Observe: $d_{WP}(\mathcal{S}_\sigma, \mathcal{S}_\tau) = 0$

iff $i(\sigma, \tau) = 0$

Thm (Wolpert)

$\exists\, \delta_0 > 0$ s.t. if $i(\sigma, \tau) \neq 0$

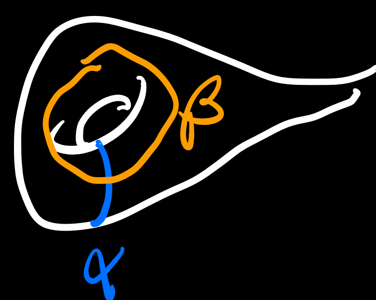
then $d_{WP}(\mathcal{S}_\sigma, \mathcal{S}_\tau) \geq \delta_0$

Questions: Is there a top. description of closest strata?

What is an explicit value for δ_0

Answer both.

Consider $T = S_{1,1}$



$\alpha, \beta \quad i(\alpha, \beta) = 1$

<u>Def</u> $\delta_{11} = d_{WP}(\mathcal{S}_\alpha, \mathcal{S}_\beta)$

Simple calc shows

$$6.59 \le \delta_{11} \le 6.63$$

Thm Let ρ_σ, ρ_τ be two states.
Then either

1) $i(\sigma,\tau) = 0$ & $d_{wp}(\rho_\sigma, \rho_\tau) = 0$

2) $i(\sigma,\tau) = 1$ & $d_{wp}(\rho_\sigma, \rho_\tau) = \delta_{11}$

3) $i(\sigma,\tau) > 1$ & $d_{wp}(\rho_\sigma, \rho_\tau) > 7.6$

Systoles

$\ell_{sys}(X)$ = length of smallest geod in X

Thm $\exists\ c : (0,\infty) \to (0,1)$ s.t.

$$c(\ell_{sys}(X)) \leq \frac{d_{WP}(X, \partial \mathrm{Teich}(S))}{\sqrt{2\pi \ell_{sys}(X)}} \leq 1$$

Furthermore $c \geq .94$.

Function $\sqrt{2\pi \ell_{sys}(X)}$

very close to dist to ∂.

InRadius

$$\mathrm{InRad}(S) = \max_X d_{WP}(X, \partial \mathrm{Teich}(S))$$

Studied by Brock-Bromberg, Wu.

Def

$$\mathrm{sys}(S) = \max_{X} \ell_{\mathrm{sys}}(X)$$

Corollary

$$c(\mathrm{sys}(X)) \leq \frac{\mathrm{InRad}(S)}{\sqrt{2\pi\, \mathrm{sys}(S)}} \leq 1$$

Furthermore

$$\lim_{g\to\infty} \frac{\mathrm{InRad}(S_{g,n})}{\sqrt{2\pi\, \mathrm{sys}(S_{g,n})}} = 1$$

Outline of Proof

1. Obtain effective bounds on $\|\nabla \ell_\alpha\|$
2. Use fact that $\overline{\mathrm{Teich}(S)}$ is CAT(0) to prove

if $i(\sigma_1, \tau) = i(\sigma_2, \tau) = 0$

then

$$d_{WP}(\rho_{\sigma_1}, \rho_{\sigma_2}) = d_{WP}(\rho_{\sigma_1 \cup \tau}, \rho_{\sigma_2 \cup \tau})$$

Using a formula of Riera Wolpert proved

<u>Thm</u> (Wolpert) Let ℓ_α, ℓ_β be length fns of simple disjoint curves. Then

$$\frac{2}{\pi}\ell_\alpha(X)\delta^\alpha_\beta \le \langle \nabla\ell_\alpha, \nabla\ell_\beta\rangle$$

$$\le \frac{2}{\pi}\ell_\alpha\delta^\alpha_\beta + O(\ell_\alpha(X)^2\ell_\beta(X)^2)$$

We prove

<u>Thm</u> Let α, β disjoint $\ell_\alpha(X) \leq \ell_\beta(X)$

$\frac{2}{\pi} \ell_\alpha(X) f_\beta^\alpha \leq \langle \nabla \ell_\alpha, \nabla \ell_\beta \rangle$

$\leq \frac{2}{\pi} \ell_\alpha f_\beta^\alpha + \frac{8}{3\pi^2} \ell_\alpha \sinh \ell_\alpha/2 \sinh^2 \ell_\beta/2$

Let $\qquad K(a,b) = \int_a^b \frac{dt}{\sqrt{\frac{2t}{\pi}}}$

$$H(a,b) = \int_a^b \frac{dt}{\sqrt{\frac{2t}{\pi} + \frac{8t}{3\pi^2} \sinh^3 t/2}}$$

Def $\quad f_\alpha^a = \ell_\alpha^{-1}(a)$

Lemma if $X \in \rho_\alpha^a$

$$|H(a,b)| \leq d_{wp}(\rho_\alpha^a, \rho_\alpha^b)$$

$$\leq d_{wp}(X, \rho_\alpha^b) \leq |K(a,b)|$$

Pf Choose X & take gradient path γ

$$L(\gamma) = \int |\gamma'(t)| \, dt$$

$$\gamma'(t) = \nabla \ell_\alpha(t)$$

$$s = \ell_\alpha(\gamma(t)) \quad s' = d\ell_\alpha(\gamma'(t))$$

$$ds = \langle \nabla \ell_\alpha, \gamma'(t) \rangle dt = \|\nabla \ell_\alpha\|^2 dt$$

$$L(\gamma) \leq \int_a^b \frac{ds}{\sqrt{\frac{2s}{\pi}}} = K(a,b)$$

Now choose γ any curve

$s = \ell_\alpha(\gamma(t))$, $ds \leq |\nabla \ell_\alpha| |\gamma'(t)| \, dt$

$$L(\gamma) \geq \int_a^b \frac{ds}{\frac{2}{\pi}(s + f(s))} = H(a,b).$$

How this helps

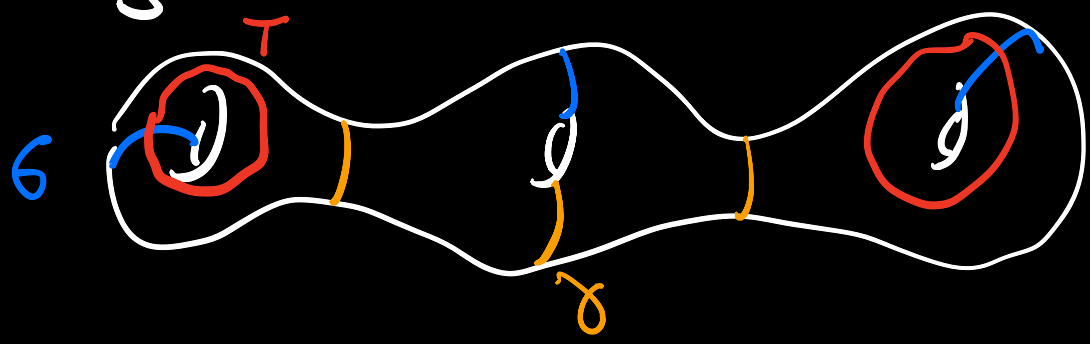
Let $i(\sigma, \tau) \neq 0$ &

if $i(\alpha, \tau) = 0$ or I

and $i(\beta, \sigma) = 0$ or I

$\forall \; \alpha \in \sigma, \beta \in \tau$

Using CAT(0) prop



$$d(\rho_\sigma, \rho_\tau) = \sqrt{k}\,\delta_{1,1} \geq 9.29.$$

for $k \geq 2$.

Otherwise

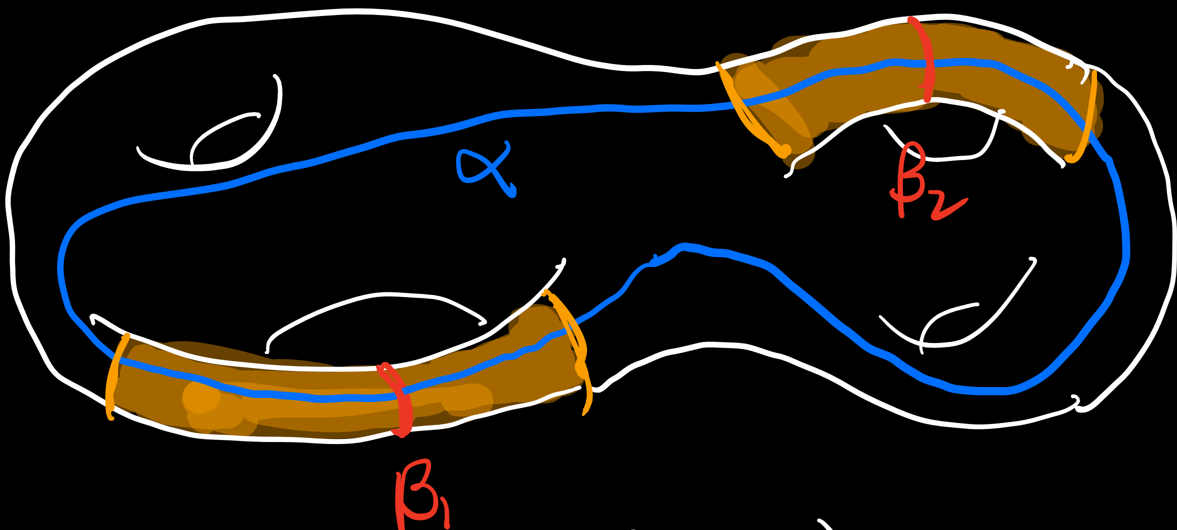
$\exists\ \alpha \quad \beta_1, \beta_2 \ (\beta_1 = \beta_2)$

$i(\alpha, \beta_1 \cup \beta_2) \geq 2$

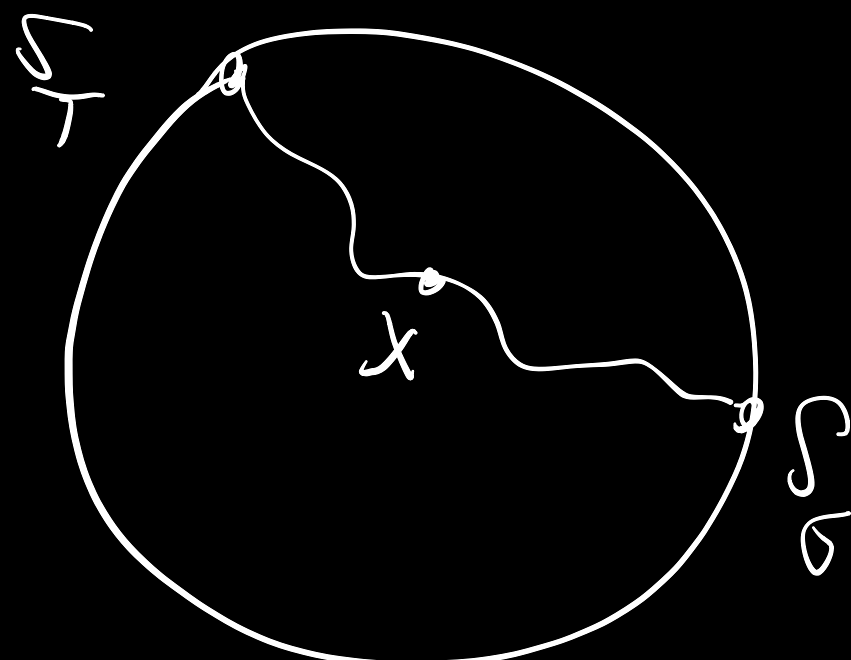
let $c(t)$ path from ρ_σ, ρ_τ

then $\exists\ c(t_0) = X$

$\max(\ell_{\beta_1}(X)) = 2\varepsilon_2$ margulis const



$$\ell_\alpha(X) \geq 4\varepsilon_2$$



$$d(X, \rho_\sigma] \geq H(0, 4\varepsilon)$$
$$d(X, \rho_T) \geq H(0, 2\varepsilon)$$

$\Rightarrow$ c has length

$$\geq H(0, 2\varepsilon) + H(0, 4\varepsilon)$$

$$\geq 7.6$$

Lagniappe (if time)

For systole bounds

Take $X \in \mathrm{Teich}(S)$

Now parametrize path

by $s = \ell_{sys}(\gamma(t))$

Note for systole

a systole has

collare at least $\ell_\alpha/2$
wide.

Gives better bound on $|\nabla \ell_{sys}|$

Also follows that

$\sqrt{\ell_{sys}}$ is $\frac{1}{2}$ Lipschitz

Wu proved K-Lipschitz
and indep showed similar
explict bd in recent paper

Sketch of Proof of $\|\nabla \ell_\alpha\|$

Follows app. of Wolpert.

Riera Formula gives

$$\langle \nabla \ell_\alpha, \nabla \ell_\beta \rangle = \frac{2}{\pi}\left(\ell_\alpha \delta_\beta^\alpha + \sum_{i=1}^{\infty} R(t_i) \right)$$

$$R(t) = t \log\left|\frac{t+1}{t-1}\right| - 2$$

t_i enumeration of cosh of lengths between lifts of α & β.

<u>Fact</u> $R(\cosh t) \cong a e^{-2t}$

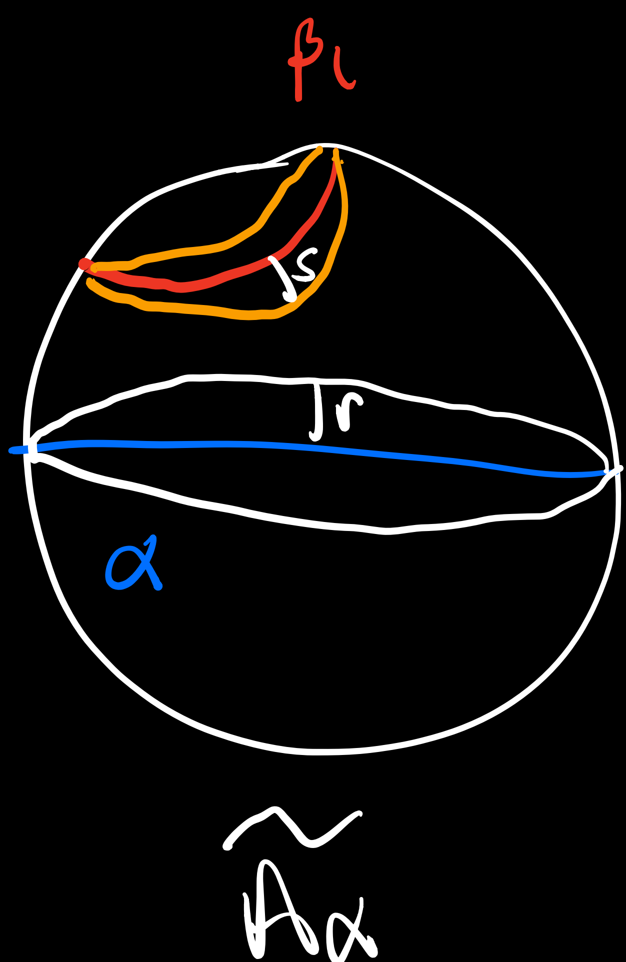
Gives bound

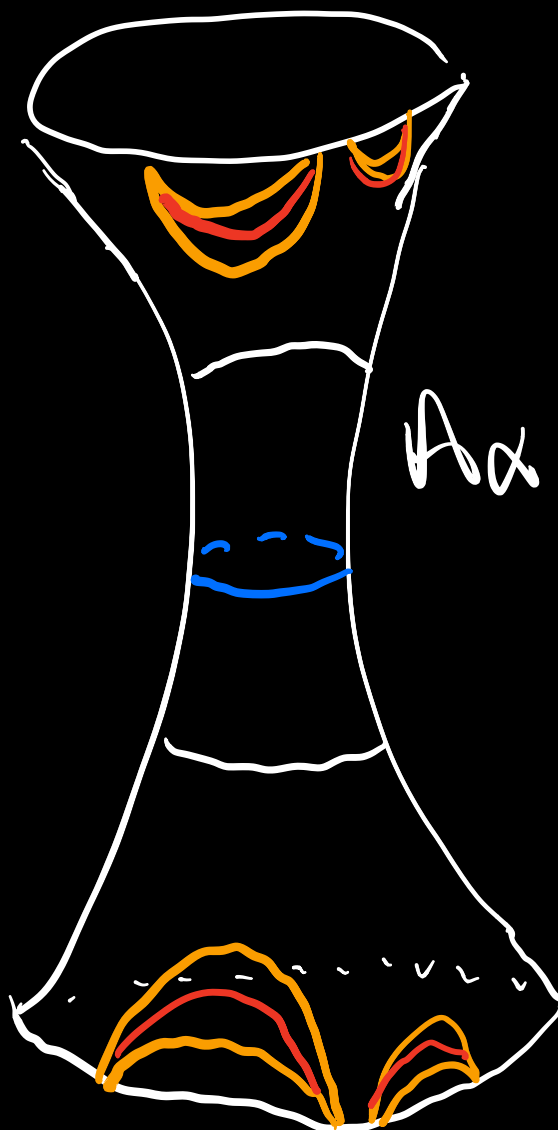
$$\sum_i R(\cosh t_i) \leq a \sum e^{-2t_i}$$

Let α, β have disjoint neighborhoods of radius r, s respectively.

Now consider annular cover A_α of α

Enumerate the lifts of β by β_i and consider $N(\beta_i, s)$ disjoint from $N(\alpha, r)$ in $\tilde{A}_\alpha$





Then letting $t = d(z, \alpha)$

$$\sum_{N(\beta_i, s)} \int e^{-2t} \, dA \le \int_{A_\alpha - N(\alpha, r)} e^{-2r} \, dA$$

$$= l_\alpha \left(e^{-r} + \frac{e^{-3r}}{3} \right)$$

Mean Value Property

$$e^{-2t_i} \le \frac{1}{g(s)} \int_{N(\beta_i, s)} e^{2t}$$

where

$$g(s) = 2\tan^{-1}(\sinh s)\cosh^2 s + 2\sinh s$$

Combining we get

$$\sum e^{-2t_i} \le \frac{\ell_\alpha \left(e^{-r} + \frac{e^{-2r}}{3}\right)}{2\tan^{-1}(\sinh s)\cosh^2 s + 2\sinh s}$$

Note can replace r, s by collar lemmas

$$r = \sinh^{-1}\left(\frac{1}{\sinh \ell_\alpha/2}\right)$$

$$s = \sinh^{-1}\left(\frac{1}{\sinh \ell_\beta/2}\right)$$