Left orderable lattices in semisimple Lie groups

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September 21, 2020

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A group Γ is *left-orderable* if it admits a total order which is invariant by left multiplications.

$$\forall f, g, h \in \Gamma : \qquad \text{If } f < g \text{ then } hf < hg$$

A folklore result

A countable group Γ is left-orderable iff it acts faithfully on the real line by orientation preserving homeomorphisms.

$$\Gamma \hookrightarrow \mathrm{Homeo}^+(\mathbb{R})$$

If $p \in \mathbb{R}$ is a free orbit (i.e. $\forall g \in \Gamma, \quad g(p) \neq p$), then we can define:

$$h <_p g \qquad \text{if} \quad h(p) < g(p).$$

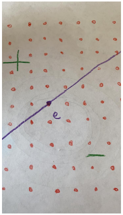
Left-orderable groups:

1. $\mathbb{Z}^n$, $\mathbb{F}_n$.
2. Braid groups. Some MCG's of surfaces. RAAG's.
3. Thompson's group F (consist of piecewise homeomorphisms of an interval)
4. Many more...

Non left-orderable groups:

1. Groups with torsion.
2. $\Gamma = \langle a, b |\ \ ab^7ab^{13}ab = e,\ ab^{-3}a^{-3}b = e,\ a^{-7}ba^{-2}b^3 = e,\ a^{-5}b^{-7}a^{-3}b^{-4} = e\rangle$.
3. Random groups. (Orlef, 2014) (Unknown for actions in the circle)
4. $SL_n(\mathbb{Z})$, when $n \geq 3$. (Witte-Morris, 1994)
5. It is unknown whether there exists an orderable group with property T.

Orders in $\mathbb{Z}^2$:



Orders in $\mathbb{F}_2$: There are many more orders (Super-exponentially many when looking at balls in the Cayley graph).

I will discuss the left-orderability of irreducible lattices in semi-simple Lie groups.

Notation: G is a Lie group, $G = \mathrm{Isom}(X)$, where X is the associated symmetric space. Γ is a lattice if $\mathrm{vol}(G/\Gamma) < \infty$.

Rank: The real rank of G is the largest n such that euclidean $\mathbb{R}^n$ embeds in X. Higher ranks means Rank(G) ≥ 2. For $G = SL_n(\mathbb{R})$, $\mathrm{Rank}(G) = n - 1$.

Hyperbolic spaces, $G = \mathbf{SO}(n, 1)$:

Fundamental groups of hyperbolic surfaces are left-orderable.

A conjecture of Boyer-Gordon-Watson, relates left-orderability of fundamental groups of 3-manifolds with taut foliations and Floer homology. See a lecture of Nathan Dunfield on his webpage.

The fundamental group of a hyperbolic 3-manifold is virtually left orderable.

Other rank one symmetric spaces:

An example of a RFRS lattice in complex hyperbolic plane by Agol-Stover is left-orderable. No examples in quaternionic hyperbolic, or Cayley plane are known to be left-orderable.

Higher rank symmetric spaces

Zimmer program: Every smooth action on a manifold of an irreducible lattice in higher rank comes from a nice algebraic construction.
Example: $SL_n(\mathbb{Z})$ acts in $\mathbb{R}^n$ linearly and projectively in $\mathbb{P}^{n-1}$. Zimmer program says $\Phi : SL_n(\mathbb{Z}) \to \mathrm{Diff}(M^{n-1})$, then $M = \mathbb{P}^{n-1}$ and action is standard.

For Homeo(M) very few things are known, Homeomorphisms are dynamically difficult.

Our main result concerns irreducible lattices in higher rank:

An irreducible lattice Γ in a connected semi-simple Lie group G of rank at least two is left-orderable iff Γ is torsion free and there exists a surjective morphism $G \to PSL(2, \mathbb{R})$.

- Dave Witte Morris and Witte Morris-Lifschitz proved this theorem for most (if not all) non-uniform lattices.

Example 1: $SL_3(\mathbb{Z})$ is not left-orderable.

Example 2: $SL(2, \mathbb{Z}(\sqrt{2}))$ embeds as a lattice in $SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$ via

$$A \to (A, \sigma(A)),$$

where $\sigma(a + b\sqrt{2}) = a - b\sqrt{2}$. Therefore $SL(2, \mathbb{Z}(\sqrt{2}))$ is not left-orderable.

Example 3: Passing to the universal covering in example 2 one gets a left-orderable lattice of higher rank.

Remark: Margulis showed all lattices in higher rank are arithmetic. So our theorem is mainly about groups similar to example 2.

A theorem of Ghys (1999):

If Γ is a lattice in a connected semi-simple Lie group G of rank at least two and $\Gamma \to \text{Homeo}^+(\mathbb{S}^1)$ is an action, then:

1. Either Γ has a finite orbit on $\mathbb{S}^1$.
2. Or there exists a surjective morphism $G \to PSL(2, \mathbb{R})$.

- This result was also proven by Burger-Monod around the same time for many lattices. Navas and Rezhnikov proved that any group with property T do not act smoothly in $\mathbb{S}^1$. Ghys Theorem was generalized by Bader-Furman for some non-linear groups.

Strategy of proof: Assume G simple. Argue by contradiction. Assume Γ acts in $\mathbb{R}$ minimally.

Goal: Show Γ preserves a measure on $\mathbb{R}$. This implies Γ is conjugated to an action by translations. $\Gamma \to \mathbb{Z}$, contradiction.

Suspension space (As in Nick Miller's talk):

$$Y := (G \times \mathbb{R}) \Big/ \Gamma \quad \text{with} \quad (g, t) \sim (g\gamma^{-1}, \gamma(t)),\ \gamma \in \Gamma$$

- Y is an $\mathbb{R}$-bundle over G/Γ. G acts on Y.
- Γ preserves a measure in $\mathbb{R}$ iff G preserves a measure on Y.

Stiffness 1: Construct a G-stationary measure on Y and show is G-invariant.

Stiffness 2: Construct a P-invariant measure on Y and show is G-invariant.

Both properties are equivalent by Furstenberg correspondence.

We take* P-invariant measure on Y and show is G-invariant.

Philosophy: Higher rank abelian (hyperbolic) actions have rigidity. Understand dynamics of A-action in Y and show G-invariance.

Remark 1: This strategy was used in work of Brown,Rodriguez-Hertz,Wang (2014) about stiffness of actions of lattices. This was later applied by Brown, Fisher, Hurtado in the solution of Zimmer's conjecture (2016).

Remark 2: Our method follows same philosophy but avoids use of entropy and Ledrappier-Young formula.

Big problems: $\mathbb{R}$ is not compact. Action is not smooth.

Theorem (Deroin's space of almost-periodic actions (2011))

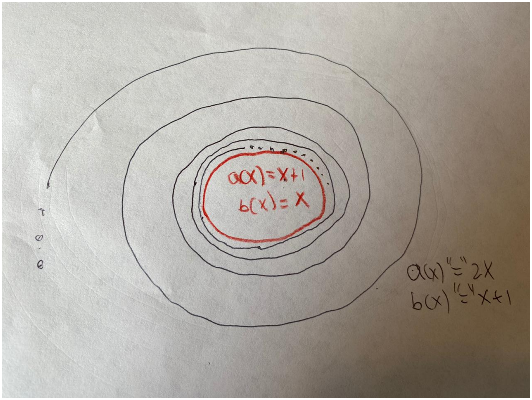
For a left orderable group Γ, there exists a compact space D with a one dimensional lamination such that:

1. *Γ acts on D and preserve each leaf.*
2. *The action is Lipschitz in each one dimensional leaf.*
3. *Any action* of Γ without a discrete orbit in $\mathbb{R}$ is conjugate to the action of Γ in a leaf of D.*

Warning: D is in general infinite dimensional and its size is related to the possible left-orders of Γ.
Remark: D is related to space of orders constructed* by Ghys.

Remark 1: For Γ lift of action by homeomorphisms of $\mathbb{S}^1$, D contains a copy of $\mathbb{S}^1$.

Example 1: For $\Gamma = \{a, b | aba^{-1} = b^2\}$.



Example 2: For $\Gamma = \mathbb{Z}^2$, D consist of actions by translations. D can be taken topologically to be $\mathbb{S}^1 \times \mathbb{S}^1$.

Some other applications of D:

1. A left orderable, amenable group has surjection to $\mathbb{Z}$. (Witte-Morris).
2. Understanding of Hyde-Lodha 's example of f.g. simple left orderable group. (Triestino-Matte Bon)
3. Rigidity of actions of Thompson's groups and other related work. (Rivas, Matte Bon, Lodha, Triestino).

Random walks by homeomorphisms of $\mathbb{R}$:

Suppose μ is a finitely supported, symmetric measure on Γ. Assume Γ fixed point free. Fix $p \in \mathbb{R}$. Consider the random walk:

$$X_n(p) = g(X_{n-1}(p))$$

g is chosen as determined by μ.

What happen as $n \to \infty$?

Theorem (Deroin-Kleptsyn-Navas-Parwani (2012))

1. *For all* $p \in \mathbb{R}$, $\limsup X_n(p) = \infty$ *and* $\liminf X_n(p) = -\infty$ *almost surely.*
2. *There exists a stationary Radon measure in* $\mathbb{R}$. *(unique* for minimal action).*
3. *Under necessary assumptions**: For all* $p, q \in \mathbb{R}$ $\lim X_n(p) - X_n(q) = 0$.

DNKP Theorem implies that up to conjugation, Lebesgue is stationary: For all $x, y \in \mathbb{R}$, $x - y = \sum \mu(\gamma)(\gamma(x) - \gamma(y))$, moreover:

1. **Lipschitz:** $|\gamma(x) - \gamma(y)| \leq \frac{1}{\mu(\gamma)}|x - y|$,
2. **Bounded displacement and non-triviality**:

$$\forall x, \quad \frac{1}{C_\mu} \leq \sum \mu(\gamma)|\gamma(x) - x| \leq C_\mu$$

3. **Harmonicity:** $\forall x,\ x = \sum \mu(\gamma)\gamma(x)$.

$$D := \{(\Phi, p) | p \in \mathbb{R},\ \ \Phi : \Gamma \to \mathrm{Homeo}^+(\mathbb{R}) \text{ satisfying } 1), 2) \text{ and } 3)\}/\sim$$

The equivalence relation $\sim$ is defined by translations: $(\Phi, p) \sim (T^t \Phi T^{-t}, p + t)$.
There is an $\mathbb{R}$-flow in D sending (Φ, p) to $(\Phi, p + t)$.

Thank you and have a nice week.

Ideas of proof of main theorem Let $X = (G \times D)/\Gamma$ be the suspension space for the Γ action on D. X is a G-space. Fix a maximal compact subgroup $K \subset G$, and let m_G be a probability measure on G which is

- absolutely continuous wrt Haar.
- invariant by left and right multiplications by K, and
- symmetric.

A general machinery shows that there exists on X a measure m_X which is m_G-stationary, namely which satisfies the convolution equation

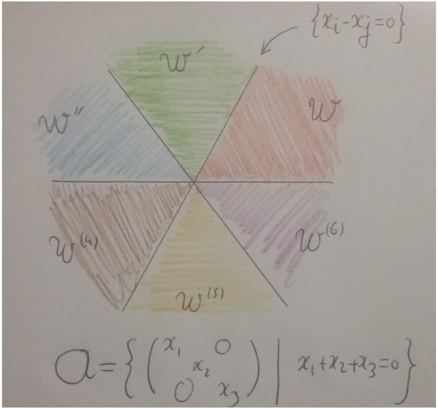
$$m_G \star m_X = \int g_* m_X \; m_G(dg) = m_X.$$

Our goal is to establish that m_X is indeed G-invariant; we construct D, X and m_X are constructed in such a way that m_X is ergodic and conditionals measures along leafs of D are abs. continuous with respect to Lebesgue. For constructing D, we choose μ in Γ a dicretization probability measure for the Brownian motion in the symmetric space $K\backslash G$. (G/P is the poisson boundary of (Γ, μ)).

Weyl chambers Consider the case $G = \mathrm{SL}(3,\mathbb{R})$. We set $K = \mathrm{SO}(3,\mathbb{R})$, and let $A \subset G = \mathrm{SL}(3,\mathbb{R})$ be the subgroup of diagonal matrices with positive coefficients. Each $a \in \mathrm{lie}(A) \simeq \mathbb{R}^2$ determines a solvable subgroup $P^a = AN^a$, where N^a is the strong unstable foliation of a:

$$N^a := \{b \in G \mid e^{ta} b e^{-ta} \to_{t \to -\infty} e_G\}.$$

For generic a's, there are only six possibilities for the N^a's, which defines a decomposition of A into six Weyl chambers:



$P^{\mathcal{W}}$-invariant measures

For each Weyl chamber $\mathcal{W}$, we have the Iwasawa decomposition $G = KP^{\mathcal{W}}$. Applying Furstenberg's Poisson formula to the function $g \mapsto g_* m_X$, which is harmonic and bounded (since m_X is stationary), one proves that:

There exists a unique probability measure $m_X^{\mathcal{W}}$ on X which satisfies

- *$m_X^{\mathcal{W}}$ is $P^{\mathcal{W}}$-invariant and $P^{\mathcal{W}}$-ergodic,*
- *the K-average of $m_X^{\mathcal{W}}$ wrt the normalized Haar measure on K equals m_X.*

Global contraction property

The lamination defined by the flow T on the quasi-periodic space Z produces a one dimensional oriented lamination $\mathcal{L}$ on the suspension space X, which is invariant by the G-action.

We say that an element $a \in \mathrm{lie}(A)$ has the *global contraction property wrt some probability measure m on X* if for m-a.e. $x \in X$, the flow associated to a contracts globally the leaf $\mathcal{L}(x)$ in the sense that

$$d(e^{ta}(y), e^{ta}(z)) \to_{t \to +\infty} 0 \text{ for every } y, z \in \mathcal{L}(x).$$

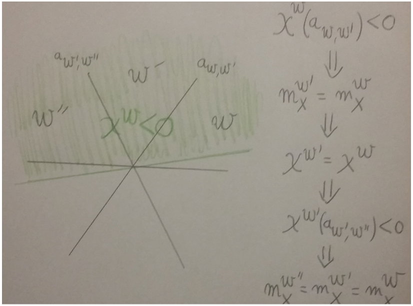
Lyapunov exponents

For each Weyl chamber $\mathcal{W}$, there exists an open half-space in lie(A) consisting of elements whose exponential have the global contraction property wrt to $m_X^{\mathcal{W}}$. Moreover, this half-space intersects the interior of $\mathcal{W}$.

This half-space is determined by a Lyapunov exponent functional being negative. The Lyapunov exponent is the exponential rate of the derivative in the direction of $\mathcal{L}$ of an element of A. It is linear functional in lie(A) and is denoted by $\chi^{\mathcal{W}}$: lie$(A) \to \mathbb{R}$.

Propagating invariance

Assume that $\mathcal{W}, \mathcal{W}'$ are two adjacent Weyl chambers, and denote by $a^{\mathcal{W},\mathcal{W}'}$ a non zero element in $\mathcal{W} \cap \mathcal{W}'$. Assume that the flow a has the global contraction property wrt $m_X^{\mathcal{W}}$. Then $m_X^{\mathcal{W}} = m_X^{\mathcal{W}'}$.



Idea of the proof main Lemma: Let a be an element of $\mathcal{W} \cap \mathcal{W}'$. Asume $m_X^{\mathcal{W}}$, $m_X^{\mathcal{W}'}$ ergodic. We show there are two Birkhoff generic points x_1, x_2 for $m_X^{\mathcal{W}}$ and $m_X^{\mathcal{W}'}$ with almost the same ergodic averages.
There is a nice relation between $m_X^{\mathcal{W}}$ and $m_X^{\mathcal{W}'}$, they are related via: $k^* m_X^{\mathcal{W}}$ and $m_X^{\mathcal{W}'}$ for $k \in K$. (k is an element of the Weyl group). This allow us to find x_1, x_2 generic in the same $(G \times \mathbb{R})/\Gamma$ leaf of X. As both measures are N_a-invariant, one can change the a-future of $\pi_{G/\Gamma}(x_1)$ to almost coincide with the future of $\pi_{G/\Gamma} x_2$. More formally, there exists $n_1, n_2 \in N_a$ such that:

$$\lim_{t \to \infty} d_{G/\Gamma}(e^{ta} n_1 \pi_{G/\Gamma}(x_1), e^{ta} n_2 \pi_{G/\Gamma}(x_2)) < \epsilon$$

Using the global contraction property we have $\lim d_X(e^{ta} n_1 x_1, e^{ta} n_2 x_2) < \epsilon$ and we are done because $n_1 x_1$ and $n_2 x_2$ can be chosen Birkhoff generic.

Thank you!