Reading group on Will Johnson's dp-finite fields

Modular lattices and cube-rank (9.1-9.3 in dp-finite fields I)

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The goal

Introduce the cube rank on a modular lattice and show it is the “minimal subadditive rank”.

To that end

- introduce the notion of modular lattice;
- define a relative independence relation on modular lattices (9.1-9.2 [DpI]);
- introduce cubes, the cube rank, and an abstract notion of rank on modular lattices;
- show (under a finiteness assumption) that the cube rank is minimal among ranks in modular lattices (9.3, [DpI]).

Lattices

Definition

A **lattice** is a partially ordered set $(P, \leqslant)$ in which every two elements $x, y \in P$ have a supremum (join) denoted by $x \vee y$, and an infimum (meet) denoted by $x \wedge y$.

Alternatively, one can define a lattice as an algebraic structure $(P, \vee, \wedge)$ (subject to natural axioms for $\vee$ and $\wedge$) and recover the order by

$$x \leqslant y \Leftrightarrow x = x \wedge y \Leftrightarrow x \vee y = y.$$

A sublattice is a substructure with respect to the algebraic definition (i.e. a subset closed under join and meet).

Lattices

Examples:

- Any linear order $(P, \leqslant)$ is a lattice ($x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$).
- Given a set A, $(\mathcal{P}(A), \subseteq)$ is a lattice ($A \wedge B = A \cap B$, $A \vee B = A \cup B$).
- If cl is a closure operator on a set M, then the set of cl-closed subsets of M ordered by inclusion forms a lattice ($A \wedge B = A \cap B$ and $A \vee B = \mathrm{acl}(A \cup B)$).
- If $(G, +)$ is a group, then $\mathrm{Sub}(G)$ the subgroups of G ordered by inclusion is a lattice ($A \wedge B = A \cap B$ and $A \vee B = \langle A \cup B \rangle$). If $(G, +)$ is abelian and $(G, +, \ldots)$ is a first order expansion, then the definable subgroups of G form a sublattice of $\mathrm{Sub}(G)$.
- If R is a ring and M is an R-module, then $(\mathrm{Sub}_R(M), \subseteq)$ is a lattice ($A \wedge B = A \cap B$ and $A \vee B = A + B$).

Modular lattices

Definition

A lattice $(P, \leqslant)$ is **modular** is it satisfies for all $a, b, x \in P$

$$a \leqslant b \Rightarrow (x \vee a) \wedge b = (x \wedge b) \vee a.$$

Examples:

- Any linear order $(P, \leqslant)$ is a modular lattice.
- $(\mathcal{P}(A), \subseteq)$ is a modular lattice (it is even distributive).
- if cl is a closure operator on A, then the lattice of cl-closed sets is not necessarily a modular lattice.
- If G is an abelian group, then $(\mathrm{Sub}(G), \subseteq)$ is modular (and hence also the sublattice of definable subgroups). The same holds for $(\mathrm{Sub}_R(M), \subseteq)$.

Modular lattices

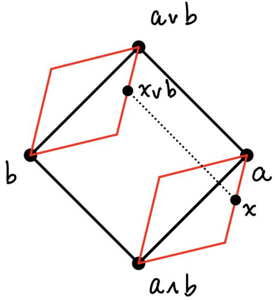
Let $(P, \leqslant)$ be a lattice. Given $a, b \in P$, the interval $[a, b]$ is defined is as $\{x \in P : a \leqslant x \text{ and } x \leqslant b\}$. Every interval is sublattice of P.

Theorem

A lattice $(P, \leqslant)$ is modular if and only if for all $a, b \in P$ the map

$$\begin{aligned} [a \wedge b, a] &\to [b, a \vee b] \\ x &\mapsto x \vee b \end{aligned}$$

is a lattice isomorphism with inverse given by $x \mapsto a \wedge x$.



Independence

From now on we let $(P, \leqslant)$ be a modular lattice. We also set $[n] := \{1, \ldots, n\}$ for $n \in \mathbb{N}$ (with $[0] = \emptyset$). The following definition is equivalent but slightly different that the one given in [DpI].

Definition

- Suppose P has a minimal element $\bot$. A sequence $(a_1, \ldots, a_n)$ of P is said to be **independent** if for every k

$$a_k \wedge \bigvee_{i \neq k} a_i = \bot.$$

- (P not necessarily with minimal element) For $b \in P$, we say $(a_1, \ldots, a_n) \in P^n$ is **independent over** b if $b \leqslant a_i$ for each $i \in [n]$ and the set is independent in the sublattice $\{x \in P : b \leqslant x\}$ (b being the minimal element).

Independence

Definition

An ***n*-cube** in P is a family $\{a_S\}_{S\subseteq[n]}$ of elements of P such that $S \mapsto a_S$ is a lattice homomorphism from $\mathcal{P}([n])$ to P. A **strict *n*-cube** is an n-cube such that this homomorphism is injective. The **top** and **bottom** of an n-cube are the elements $a_\emptyset$ and $a_{[n]}$, respectively.

Proposition (Proposition 9.15, DpI)

Let b be an element of P.

1. *If $\{a_S\}_{S\subseteq[n]}$ is an n-cube with bottom b, then $a_{\{1\}}, \dots, a_{\{n\}}$ is an independent sequence over b.*
2. *This establishes a bijection from the collection of n-cubes with bottom b to the collection of independent sequences over b of length n.*
3. *If $a_1, \dots, a_n$ is an independent sequence over b, the corresponding n-cube is strict if and only if every a_i is strictly greater than b.*

Remark

A dual statement of the above proposition also holds (see Proposition 9.16 [DpI]).

Independence

A couple of words about the proof.

- Points (1) and (3) are easy.
- For point (2), injectivity is also easy and all the work is hidden showing surjectivity. Assuming $(a_1, \ldots, a_n)$ is an independent sequence over b, one sets $a_\emptyset := b$ and $a_S := \bigvee_{i \in S} a_i$ for $\emptyset \neq S \subseteq [n]$. We need to show
  (i) $a_{S \cup S'} = a_S \vee a_{S'}$
  (ii) $a_{S \cap S'} = a_S \wedge a_{S'}$

  Point (i) follows from the definition of a_S, and point (ii) follows from the following two lemmas.

Independence

Lemma (Lemma 9.7 [DpI])

Let (c_1, c_2, c_3) be independent over b. Then $c_1 = (c_2 \vee c_1) \wedge (c_3 \vee c_1)$.

Lemma (Lemma 9.6 [DpI])

Let $(a_1, \ldots, a_n)$ be independent over b and $S_1, \ldots, S_m$ be disjoint subsets of $[n]$. Then $(a_{S_1}, \ldots, a_{S_m})$ is independent over b.

The first one uses modularity and independence. The second one needs a bit of work after unravelling definitions (maybe I missed something here, but as far as I can see one needs to show on the way the following particular case of (ii) above: if $i, j \in [n]$ are different, then $a_{[n]\setminus\{i\}} \wedge a_{[n]\setminus\{j\}} = a_{[n]\setminus\{i,j\}}$).

Cube-rank and additivity

Definition

The **cube rank** of a modular lattice $(P, \leqslant)$ (written $\mathrm{rk}^{\diamond}(M)$) is the supremum of $n \in \mathbb{Z}$ such that a strict n-cube exists in P, or ∞ if there is no supremum. If $a \geqslant b$ are two elements of P, the **cube rank** $\mathrm{rk}^{\diamond}(a/b)$ is the cube rank of the sublattice $[b, a]$.

Remark

The cube rank is called the "reduced rank" in [DpI] and [DpIII] and denoted by rk_0. I followed the notation and terminology from the report.

Goal: show that the cube rank is "the minimal subadditive rank on a modular lattice".

Cube-rank and additivity

A **subadditive rank** on a modular lattice is a function assigning a non-negative integer $\mathrm{rk}(a/b)$ to every pair $a \geq b$, satisfying the following axioms

1. $\mathrm{rk}(a/b) = 0$ if and only if $a = b$.
2. If $a \geq b \geq c$, then

$$\begin{aligned} \mathrm{rk}(a/c) &\geq \mathrm{rk}(a/b) \\ \mathrm{rk}(a/c) &\geq \mathrm{rk}(b/c) \\ \mathrm{rk}(a/c) &\leq \mathrm{rk}(a/b) + \mathrm{rk}(b/c). \end{aligned}$$

3. If a, b are arbitrary, then

$$\begin{aligned} \mathrm{rk}(a/a \wedge b) &= \mathrm{rk}(a \vee b/b) \\ \mathrm{rk}(b/a \wedge b) &= \mathrm{rk}(a \vee b/a) \\ \mathrm{rk}(a \vee b/a \wedge b) &= \mathrm{rk}(a/a \wedge b) + \mathrm{rk}(b/a \wedge b). \end{aligned}$$

A **weak subadditive rank** is a function satisfying the axioms other than (1).

Cube-rank and additivity

Goal: show that the cube rank is "the minimal subadditive rank on a modular lattice". This corresponds formally to

Proposition

Let $(P, \leq)$ *be a modular lattice.*

1. *If the cube rank* $\mathrm{rk}^\diamond(a/b)$ *is finite for all pairs* $a \geq b$*, then* $\mathrm{rk}^\diamond$ *is a (non-weak) subadditive rank.*
2. *If there is a subadditive rank* rk *on* P*, then* $\mathrm{rk}^\diamond(a/b) \leq \mathrm{rk}(a/b) < \infty$ *for all* $a \geq b$*.*

Cube-rank and additivity

Some comments on the proof: Part (2) goes by induction and it's not difficult. Most work goes on part (1), and particularly to show

$$\mathrm{rk}^{\diamond}(a/c) \leq \mathrm{rk}^{\diamond}(a/b) + \mathrm{rk}^{\diamond}(b/c)$$
$$\mathrm{rk}^{\diamond}(a \vee b/a \wedge b) = \mathrm{rk}^{\diamond}(a/a \wedge b) + \mathrm{rk}^{\diamond}(b/a \wedge b).$$

The first one follows form the following lemma

Lemma (Lemma 9.22, [DpI])

Let $x \leq y \leq z$ be three elements of P. If there is a strict n-cube $\{a_S\}_{S \subseteq [n]}$ in $[x, z]$, then we can write $n = m + \ell$ and find a strict m-cube in $[x, y]$ and a strict ℓ-cube in $[y, z]$.

The proof uses modularity and the above correspondence between n-cubes and independent sequences of length n (and its dual statement).

Cube-rank and additivity

The condition

$$\mathrm{rk}^{\diamond}(a \vee b / a \wedge b) = \mathrm{rk}^{\diamond}(a / a \wedge b) + \mathrm{rk}^{\diamond}(b / a \wedge b).$$

is achieved using the following lemma:

Lemma (Lemma 9.18, [DpI])

1. *Let P_1 and P_2 be two modular lattices. If the reduced rank of P_1 is at least n and the reduced rank of P_2 is at least m, then the reduced rank of $P_1 \times P_2$ is at least $n + m$.*
2. *If P_1 is a modular lattice and P_2 is a sublattice, the reduced rank of P_1 is at least the reduced rank of P_2.*
3. *If a, b are two elements of a modular lattice $(P, \leq)$, then there is an injective lattice homomorphism*

$$\begin{aligned} [a \wedge b, a] \times [a \wedge b, b] &\to [a \wedge b, a \vee b] \\ (x, y) &\mapsto x \vee y. \end{aligned}$$

4. *If a, b are two elements of a modular lattice $(P, \leq)$ such that $\mathrm{rk}^{\diamond}(a / a \wedge b) \geq n$ and $\mathrm{rk}^{\diamond}(b / a \wedge b) \geq m$, then $\mathrm{rk}^{\diamond}(a \vee b / a \wedge b) \geq n + m$.*

References I

Will Johnson, **Dp-finite fields i: infinitesimals and positive characteristic**, 2019, arXiv:1903.11322 [math.LO].