Reading group on Will Johnson's Dp-finite fields papers

Dp-finite I §9.3, Dp-finite III §6.1

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Remarks

There are some "Speculative Remarks" in 9.3 of "Dp-finite fields I" that I have not included in today's presentation. Will writes that they provide some motivation, but they are not needed for the main line of the proof.

I plan to present Proposition 9.31 of "Dp-finite fields I" and §6 of "Dp-finite fields III".

Another interesting remark from Will by email: "I learned recently that "breadth" is the correct name in lattice theory for what I've been calling reduced-rank/cube-rank."

§6 Dp-finite III

Proposition (6.3)

Let $M = (|M|, \vee, \wedge, \perp, \top)$ be a bounded modular lattice and $n > 1$ any integer. The following are equivalent.

1. *There is a strict n-cube in M*
2. *There are $a_1, \dots, a_n \in M$ such that for any $1 \le i \le n$,*

$$a_1 \vee \cdots \vee a_n \neq a_1 \vee \cdots \vee \widehat{a_i} \vee \cdots \vee a_n$$

3. *There are $a_1, \dots, a_n \in M$ such that for any $1 \le i \le n$,*

$$a_1 \wedge \cdots \wedge a_n \neq a_1 \wedge \cdots \wedge \widehat{a_i} \wedge \cdots \wedge a_n$$

✓ $\underline{(3) \Rightarrow (1)}$: $b_i = \bigwedge_{j \neq i} a_j$, $c = \bigwedge_{j \geq n} a_j$. $b_i \geq c$. By assump: $b_i > c$.

$\underline{\underline{\text{cl}}}$: $b_1 \dots b_n$ indpt over c. If $i \neq k$: $b_i = b_i \wedge a_k \leq a_k$. So

$$c \leq (\underbrace{b_1}_{\leq a_k} \vee \cdots \vee \underbrace{b_{k-1}}_{\leq a_k}) \wedge b_k \leq a_k \wedge b_k = c$$

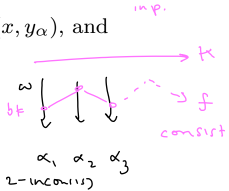
$\underline{(1) \Rightarrow (2)}$:

Burden

Definition (*inp*-patterns)

An *inp*-pattern in $p(x)$ of depth κ consists of $(a_{\alpha,i})_{\alpha<\kappa,i<\omega}$, $\phi_\alpha(x, y_\alpha)$, and $k_\alpha < \omega$ such that

- $\{\phi_\alpha(x, a_{\alpha,i})\}_{i<\omega}$ is k_α-inconsistent, for each $\alpha < \kappa$, and
- $\{\phi_\alpha(x, a_{\alpha,f(\alpha)})\}_{\alpha<\kappa} \cup p(x)$ is consistent, for any $f : \kappa \to \omega$.



Definition (burden)

Given a partial type $p(x)$, $\mathrm{bdn}(p)$ is the supremum of depths of all inp-patterns in $p(x)$. $\mathrm{bdn}(a/C) = \mathrm{bdn}(\mathrm{tp}(a/C))$.

In calculating the sup: identify every limit cardinal κ with κ^+ and insert an element κ^- directly before κ^+ in the order.

sub-multiplicativity $\mathrm{bdn}(a_i) < k_i$ for $k_i \in \omega \Rightarrow \mathrm{bdn}(a_0, \dots, a_{n-1}) < \prod_{i<n} k_i$

burden is sub-multiplicative (Chernikov, 2014)

$\boxed{\mathrm{bdn}(p) \leq \text{dp-rk}(p)}$ + [(NIP T): $\mathrm{bdn}(p) = \text{dp-rk}(p)$] (Adler, 2007)

sub-additivity (NIP T): $\mathrm{bdn}(ab) \leq \mathrm{bdn}(a) + \mathrm{bdn}(b)$

NIP T: dp-rank is sub-additive (Kaplan-Onshuus-Usvyatsov, 2013)

dp-finite $\Rightarrow$ finite burden

CKS Prop 4.5

The following result is from the 2014 Chernikov-Kaplan-Simon paper "Groups and Fields with NTP_2".

Definition

T is strong if $\mathrm{bdn}(x = x) \leq \aleph_0^-$.

A strong theory must be NTP_2.

Proposition (4.5)

Let G be a type-definable group and $(G_i \leq G : i < \omega)$ type-definable normal subgroups.

1. *If T is strong, then there is some i_0 such that $\left[\bigcap_{i \neq i_0} G_i : \bigcap_{i<\omega} G_i\right] < \infty$.*
2. *If T is of finite burden $n-1$, then there is some $n \in \omega$ and $i_0 < n$ such that $\left[\bigcap_{i \neq i_0, i<n} G_i : \bigcap_{i<n} G_i\right] \boxed{< \infty}$.*

Proposition 4.5.2 follows from 4.5.1 by sub-multiplicativity of burden.

Sps not:

Erdős-Rado

$i=0 \quad (\hat{a}_{0,j})_{j<\lambda} \in \bigcap_{k\neq 0} G_k \qquad a_{0,j_1} \neq a_{0,j_2} \mod G_0 \qquad (a_{0,j})_{j<\omega}$ preserve the 1-type, 2-type

$k = n \qquad i=1 \quad (\hat{a}_{1,j})_{j<\lambda}$

$i=2$

$\vdots$

§6 Dp-finite III, part 2

Definition

Given two type-definable groups G and H, we say that G is 00-commensurable with H ($G \approx H$) if $[G : G \cap H] < \infty$ and $[H : G \cap H] < \infty$. ✓

✓ Proposition (6.4)

$n = bdn(G)$

Let G be a definable abelian group with finite burden, and M be the lattice of type-definable subgroups of G, modulo 00-commensurability. Then M is cube-bounded; in fact $rk^{\diamond}(M) \leq bdn(G)$.

The proof uses Proposition 6.3 and CKS Proposition 4.5.2.

Let $A_1 \dots A_{n+1}$ be type-definable subgroups. $\exists i$

$$\Big[\bigcap_{\substack{j \neq i \\ j \leq n+1}} A_j : \bigcap_{j \leq n+1} A_j \Big] < \infty$$

In $\underset{M}{\overset{\shortparallel}{L^{\circ\circ}}}$, by Prop 6.3, there is not a strict $(n+1)$-cube

$\Rightarrow rk^{\diamond}(M) \leq n = bdn(G)$.

§8 of Dp-finite I

Theorem (8.4)

Let $(\mathbb{G}, +, \dots)$ be a monster-model abelian group of dp-rank $n < \omega$. There is a cardinal κ such that for any type definable subgroup $H < \mathbb{G}$, $(H : H^{00}) < \kappa$ (in any elementary extension of $\mathbb{G}$).

The proof uses CKS Proposition 4.5.2 as well as some similar ideas from the proof.

Corollary (8.7)

Let $\mathbb{M}$ be a field of finite dp-rank. There is a cardinal $\boxed{\kappa}$ such that for any small model $M \prec \mathbb{M}$ of cardinality at least κ, if J is a type-definable M-linear subspace of $\mathbb{M}^k$, then $J = J^{00}$.

*

$A \cap B = (A \cap B)^{00} = A \cap B$

J is assumed to be type-definable over a (small) set of parameters from $\mathbb{M}$.

Take κ as in Thm 8.4.

$M \prec \mathbb{M}$, $|M| \geq \kappa$ *

Fix $J \subseteq \mathbb{M}$ type-def'l M-linear subspace.

J/J^{00} v.s. / M

If nontrivial $\Rightarrow$ size $\geq \kappa$ $\Rightarrow$ contradicting index $< \kappa$.

§9 of Dp-finite I

In p. 26 of the report, Will writes that the definition $G \wedge H = (G \cap H)^{00}$ "causes too many problems". Over email, he writes: "Specifically, something goes wrong in Lemma 5.7 of Dp-finite fields V if we try to work modulo 00-commensurability."

Thus, the main line of the proof takes the following approach:

Proposition (9.31)

Let $\mathbb{M}$ be a monster-model field, possibly with additional structure, such that dp-rk$(\mathbb{M}) = n < \omega$. Let $K \prec \mathbb{M}$ be a small submodel as in Corollary 8.7. Let L_K be the modular lattice of K-linear subspaces of $\mathbb{M}$, type-definable over (small) parameter sets. Then rk$^\diamond(L_K) \leq n$.

L = type-definable subgroups of $(\mathbb{M}, +)$

$L^{00} = L/\approx$ (mod 00-commensurability)

$L_K \longrightarrow L \longrightarrow L^{00}$ so $L_K \hookrightarrow L^{00}$

thus: $\mathrm{rk}^\diamond(L_K) \leq \mathrm{rk}^\diamond(L^{00}) \leq \mathrm{bdn}(\mathbb{M}) = \text{dp-rk}(\mathbb{M}) = n$

$\uparrow$ Prop 6.4 Dp-III $\qquad$ $\uparrow$ since T is NIP

§6 Dp-finite III, part 3

Definition

A multi-valuation ring on a field K is a finite intersection of valuation rings on K.

Proposition (6.2.4 Dp II) *

Let $\mathcal{O}_1, \ldots, \mathcal{O}_n$ be pairwise incomparable valuation rings on a field K, and let $\boxed{R = \bigcap_i \mathcal{O}_i}$. Every R-submodule of K is of the form $\bigcap_i \{x \in K \mid \nu_i(x) > \Xi_i\}$ for certain cuts Ξ_i in the value groups Γ_i.

Corollary (6.7 Dp II)

If R is a multi-valuation ring on a field K, then there is a unique way to write R as a finite intersection of pairwise-incomparable valuation rings on K,

* $R = \mathcal{O}_1 \cap \cdots \cap \mathcal{O}_n$.

Lemma (6.5)

Let $R = \mathcal{O}_1 \cap \cdots \cap \mathcal{O}_n$ be an intersection of n pairwise incomparable valuation rings on a field K. Then $rk^{\diamond}(Sub_R(K)) = n$.

By Cor 6.7, no sub-intersection $= \mathcal{O}_1 \cap \cdots \cap \mathcal{O}_n$

Let $M = \mathrm{Sub}_R(K)$. By Prop 6.3, $rk^{\diamond}(M) \geq n$. Sps $rk^{\diamond}(M) > n$:

Then $\exists\, A_0 \ldots A_{n+1} \in M$: $\underbrace{A_0 \cap \cdots \cap A_n}$ $\neq A_0 \cap \cdots \cap \hat{A}_j \cap \cdots \cap A_n$ (for any j)

Prop 6.2.4: $A_j = \bigcap_{i=1}^{n} \{x \in K \mid v_i(x) > \Xi_{ij}\}$ $\Xi_i' = \max_{0 \leq j \leq n} \Xi_{ij}$; $1 \leq i \leq n$

$n < n+1$, contradiction.