Reading group on Will Johnson's Dp-finite fields papers

Dp-finite I §9.3, Dp-finite III §6.1

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Remarks

There are some “Speculative Remarks” in 9.3 of “Dp-finite fields I” that I have not included in today’s presentation. Will writes that they provide some motivation, but they are not needed for the main line of the proof.

I plan to present Proposition 9.31 of “Dp-finite fields I” and §6 of “Dp-finite fields III”.

Another interesting remark from Will by email: “I learned recently that “breadth” is the correct name in lattice theory for what I’ve been calling reduced-rank/cube-rank.”

§6 Dp-finite III

Proposition (6.3)

Let $M = (|M|, \vee, \wedge, \bot, \top)$ *be a bounded modular lattice and* $n > 1$ *any integer. The following are equivalent.*

1. *There is a strict* n*-cube in* M
2. *There are* $a_1, \ldots, a_n \in M$ *such that for any* $1 \leq i \leq n$*,*

$$a_1 \vee \cdots \vee a_n \neq a_1 \vee \cdots \vee \widehat{a_i} \vee \cdots \vee a_n$$

3. *There are* $a_1, \ldots, a_n \in M$ *such that for any* $1 \leq i \leq n$*,*

$$a_1 \wedge \cdots \wedge a_n \neq a_1 \wedge \cdots \wedge \widehat{a_i} \wedge \cdots \wedge a_n$$

Burden

Definition (*inp*-patterns)

An *inp*-pattern in $p(x)$ of depth κ consists of $(a_{\alpha,i})_{\alpha<\kappa, i<\omega}$, $\phi_\alpha(x, y_\alpha)$, and $k_\alpha < \omega$ such that

- $\{\phi_\alpha(x, a_{\alpha,i})\}_{i<\omega}$ is k_α-inconsistent, for each $\alpha < \kappa$, and
- $\{\phi_\alpha(x, a_{\alpha,f(\alpha)})\}_{\alpha<\kappa} \cup p(x)$ is consistent, for any $f : \kappa \to \omega$.

Definition (burden)

Given a partial type $p(x)$, $\mathrm{bdn}(p)$ is the supremum of depths of all inp-patterns in $p(x)$. $\mathrm{bdn}(a/C) = \mathrm{bdn}(\mathrm{tp}(a/C))$.

In calculating the sup: identify every limit cardinal κ with κ^+ and insert an element κ^- directly before κ^+ in the order.

sub-multiplicativity $\mathrm{bdn}(a_i) < k_i$ for $k_i \in \omega \Rightarrow \mathrm{bdn}(a_0, \dots, a_{n-1}) < \prod_{i<n} k_i$
burden is sub-multiplicative (Chernikov, 2014)
$\mathrm{bdn}(p) \leq \text{dp-rk}(p)$ + [(NIP T): $\mathrm{bdn}(p) = \text{dp-rk}(p)$] (Adler, 2007)

sub-additivity (NIP T): $\mathrm{bdn}(ab) \leq \mathrm{bdn}(a) + \mathrm{bdn}(b)$
NIP T: dp-rank is sub-additive (Kaplan-Onshuus-Usvyatsov, 2013)

dp-finite $\Rightarrow$ finite burden

CKS Prop 4.5

The following result is from the 2014 Chernikov-Kaplan-Simon paper "Groups and Fields with NTP_2".

Definition

T is strong if $\mathrm{bdn}(x = x) \leq \aleph_0^-$.

A strong theory must be NTP_2.

Proposition (4.5)

Let G be a type-definable group and $(G_i \leq G : i < \omega)$ type-definable normal subgroups.

1. *If T is strong, then there is some i_0 such that $\left[\bigcap_{i \neq i_0} G_i : \bigcap_{i<\omega} G_i\right] < \infty$.*
2. *If T is of finite burden, then there is some $n \in \omega$ and $i_0 < n$ such that $\left[\bigcap_{i \neq i_0, i<n} G_i : \bigcap_{i<n} G_i\right] < \infty$.*

Proposition 4.5.2 follows from 4.5.1 by sub-multiplicativity of burden.

§6 Dp-finite III, part 2

Definition

Given two type-definable groups G and H, we say that G is 00-commensurable with H ($G \approx H$) if $[G : G \cap H] < \infty$ and $[H : G \cap H] < \infty$.

Proposition (6.4)

Let G be a definable abelian group with finite burden, and M be the lattice of type-definable subgroups of G, modulo 00-commensurability. Then M is cube-bounded; in fact $rk^{\diamond}(M) \leq bdn(G)$.

The proof uses Proposition 6.3 and CKS Proposition 4.5.2.

§8 of Dp-finite I

Theorem (8.4)

Let $(\mathbb{G}, +, \ldots)$ *be a monster-model abelian group of dp-rank* $n < \omega$*. There is a cardinal* κ *such that for any type definable subgroup* $H < \mathbb{G}$*,* $(H : H^{00}) < \kappa$ *(in any elementary extension of* $\mathbb{G}$*).*

The proof uses CKS Proposition 4.5.2 as well as some similar ideas from the proof.

Corollary (8.7)

Let $\mathbb{M}$ *be a field of finite dp-rank. There is a cardinal* κ *such that for any small model* $M \prec \mathbb{M}$ *of cardinality at least* κ*, if* J *is a type-definable* M*-linear subspace of* $\mathbb{M}^k$*, then* $J = J^{00}$*.*

J is assumed to be type-definable over a (small) set of parameters from $\mathbb{M}$.

§9 of Dp-finite I

In p. 26 of the report, Will writes that the definition $G \wedge H = (G \cap H)^{00}$ "causes too many problems". Over email, he writes: "Specifically, something goes wrong in Lemma 5.7 of Dp-finite fields V if we try to work modulo 00-commensurability."

Thus, the main line of the proof takes the following approach:

Proposition (9.31)

Let $\mathbb{M}$ be a monster-model field, possibly with additional structure, such that dp-rk$(\mathbb{M}) = n < \omega$. Let $K \prec \mathbb{M}$ be a small submodel as in Corollary 8.7. Let L_K be the modular lattice of K-linear subspaces of $\mathbb{M}$, type-definable over (small) parameter sets. Then rk$^\diamond(L_K) \leq n$.

§6 Dp-finite III, part 3

Definition

A multi-valuation ring on a field K is a finite intersection of valuation rings on K.

Proposition (6.2.4 Dp II)

Let $\mathcal{O}_1, \ldots, \mathcal{O}_n$ be pairwise incomparable valuation rings on a field K, and let $R = \bigcap_i \mathcal{O}_i$. Every R-submodule of K is of the form $\{x \in K \mid \nu_i(x) > \Xi_i\}$ for certain cuts Ξ_i in the value groups Γ_i.

Corollary (6.7 Dp II)

If R is a multi-valuation ring on a field K, then there is a unique way to write R as a finite intersection of pairwise-incomparable valuation rings on K, $R = \mathcal{O}_1 \cap \cdots \cap \mathcal{O}_n$.

Lemma (6.5)

Let $R = \mathcal{O}_1 \cap \cdots \cap \mathcal{O}_n$ be an intersection of n pairwise incomparable valuation rings on a field K. Then $rk^{\diamond}(Sub_R(K)) = n$.