Dp-finite fields V, after W. Johnson

Zoé Chatzidakis

22 and 29 October 2020
MSRI - Working group

Definition 1.1. Let R be a domain, $K = Frac(R)$, $n \in \mathbb{N}$, $S \subset K$.
(1) Then S is a W_n-*set* iff whenever $a_0, \ldots, a_n \in K$, then for some i, $a_i \in \sum_{j \neq i} a_j S$.
(2) R is a W_n-ring iff it is a W_n-set as above; it suffices to check the condition for $a_0, \ldots, a_n \in R$. Equivalently, if $\mathsf{c}{-}\mathsf{rk}_R(R) = \mathsf{c}{-}\mathsf{rk}_R(K) \leq n$.

Definition 1.2. A V-*topology* is a locally bounded field topology such that for any $B \subseteq K^\times$, B^{-1} is bounded iff $K \setminus B \in \tau$.

Kowalski-Dürbaum: comes from an archimedean absolute value, or from a non-trivial valuation.

1.3. Fact from [PZ]. *Let τ be an ω-complete topology on the field K. If τ is locally bounded/locally bounded field/V-topology, then $\tau = \tau_R$ for some subring R with $Frac(R) = K$/and R is local/a valuation ring.*

Proposition 3.1 *Let R be an integral domain, $K = Frac(R) \neq R$. Then τ_R, with neighbourhood basis $\{cR \mid c \in K^\times\}$ is a Hausdorff non-discrete, locally bounded ring topology on K. Moreover, if R is a W_n-ring, then τ_R is a field topology.*
Note that if $d \in R$, then $dcR \subset cR$. The first assertions follow easily.

We know that $J(R) \neq 0$ (because $R/J(R)$ is the direct product of at most n fields), and $R \neq K$. We need to show that if I is an ideal of R there is a non-zero ideal I' of R such that

$$(1 + I')^{-1} \subseteq 1 + I.$$

Let $I' = I \cap J(R)$; then $I' \neq 0$ and one checks that if $x \in I'$, then $1 + x \in R^{\times}$, whence $-x(1 + x)^{-1} \in I$ and $(1 + x)^{-1} \in 1 + I$.

Definition 1.4. A W_n-topology on a field K is a Hausdorff non-discrete locally bounded ring topology on K such that for every neighbourhood U of 0 there is $c \in K^\times$ such that cU is a W_n-set. This is a local property.

Proposition 3.6. *Let R be a non-trivial W_n-ring with fraction field K. Then the induced field topology is a W_n-topology.*

Proof. The bounded neighbourhood R is a W_n-set.

Lemma 3.7. *Let K be a field with a W_n-topology. Suppose that K is ω-complete. Then the topology is induced by a W_n-ring R. Moreover, given any bounded set S, we may assume that $S \subseteq R$.*

Proof. Let U be a bounded neighbourhood of 0; for some $c \in K^\times$, cU is a W_n-set. Let R be the subring of K generated by $S \cup cU$. It is an increasing union of countably many bounded sets, hence by ω-completeness, it is bounded. So R is a bounded W_n-set which induces the topology.

Corollary 3.8 *Let K be a field with a ring topology τ.*
(1) τ is a W_n-topology iff (K, τ) is locally equivalent to a field with a topology induced by a W_n-ring R.
(2) If τ is a W_n-topology, then τ is a field topology.
(3) A W_n-topology is a W_m-topology for $m > n$.
(4) τ is a W_1-topology iff τ is a V-topology.

Proposition 4.1 *Let τ be a W_n-topology on $\mathbb{K}$ (a highly saturated field). Suppose there is a bounded neighbourhood $U \subseteq \mathbb{K}$ of* 0*, which is $\vee$-definable or $\wedge$-definable, and is an additive subgroup of $\mathbb{K}$. Then U is co-embeddable with a definable set, and τ is a definable topology.*

Proof. Co-embeddable: A and B are co-embeddable iff there are $c, d \in \mathbb{K}^\times$ such that $cA \subseteq B \subseteq dA$.
We may assume that U is a W_n-set. Let m be minimal such that U is W_m, and let $b_1, \ldots, b_m \in \mathbb{K}$ such that for all i, $b_i \notin \sum_{j \neq i} b_j U$.
Because U is a subgroup of $\mathbb{K}$, so are each of the finite sums above, and therefore they are open and closed for the topology.

Let S be the set of $x \in \mathbb{K}$ such that for some i, $b_i \in xU + \sum_{j \neq i} b_j U$. If U is $\vee$-definable, so is S, and $\mathbb{K} \setminus S$ is $\wedge$-definable. By definition and because U is a W_m-set, if $x \notin S$, then $x \in \sum_i b_i U$.

Claim There is $V \in \tau$ such that $V \cap S = \emptyset$.
We know that for every i, b_i is not in the closed set $\sum_{j \neq i} b_j U$; hence there is some small $V \in \tau$ such that $(b_i + V \cdot U) \cap \sum_{j \neq i} b_j U = \emptyset$. Take such a V which works for all b_i. If $x \in V \cap S$, then for some i, $b_i \in xU + \sum_{j \neq i} b_j U$, and

$$\emptyset \neq (b_i + xU) \cap \sum_{j \neq i} b_j U \subseteq (b_i + V \cdot U) \cap \sum_{j \neq i} b_j U = \emptyset.$$

We then get $V \subseteq \mathbb{K} \setminus S \subseteq \sum_i b_i U$. Now, $\sum_i b_i U$ is bounded, whence $\mathbb{K} \setminus S \in \tau \cap \tau^{\perp}$, i.e., it is co-embeddable with U. One of $\{U, \mathbb{K} \setminus S\}$ is $\vee$-definable, the other is $\wedge$-definable. This implies that U is co-embeddable with a definable D: say U is $\vee$-definable; wma $U \subseteq \mathbb{K} \setminus S$; by compactness there is a definable D with $U \subseteq D \subseteq \mathbb{K} \setminus S$.

Corollary. (Prop 4.6). *Let R be a non-trivial $\vee$-definable W_n-ring on $\mathbb{K}$. Then R is co-embeddable with a definable set D, and the topology is definable.*

If R is a $\vee$-definable W_n-ring on $\mathbb{K}$, this applies in particular to:
The integral closure $\tilde{R}$ of R;
If $\mathfrak{P}$ is a maximal ideal of R, the localization $R_{\mathfrak{P}}$.
(They are W_n because they contain R).

Theorem 4.10. *Let (K, τ) be a field with a W_n-topology.*
(1) There is at least one V-topological coarsening of τ.
(2) There are at most n such coarsenings.
(3) If (K, τ) is a definable topology (wrt to some structure on K), then every V-topological coarsening of τ is definable.

Theorem 4.10. *Let (K, τ) be a field with a W_n-topology.*
(1) There is at least one V-topological coarsening of τ.

Proof. (1) Let $D \subseteq K$ be a bounded neighbourhood of 0 which is a W_n-set, let (K^*, D^*) be a highly saturated elementary extension of (K, D). Then D^* defines a W_n-topology on K^* which is ω-complete. If R is the subring of K^* generated by D^*, then R is $\vee$-definable and bounded; sor R and D^* define the same topology on K^*. Let $\tilde{R}$ be the integral closure of R, P a maximal ideal of $\tilde{R}$; so the localization $\tilde{R}_P$ is $\vee$-definable, a valuation ring, and it therefore induces a V-topology on K^*, which is coarser than the one induced by D^*. This topology is definable (by 4.1). Hence, the same is true of (K, D): it has a definable V-topology which is coarser than the topology defined by D.

Theorem 4.10. *Let (K, τ) be a field with a W_n-topology.*
(1) There is at least one V-topological coarsening of τ.
(2) There are at most n such coarsenings.

---

(2) Let $\sigma_1, \ldots, \sigma_m$ be distinct V-topological coarsenings of τ; we want $m \leq n$.
Via an ultraproduct construction, we may assume that the topologies $\sigma_1, \ldots, \sigma_m$ are ω-complete. We know that τ is induced by a subring R, of weight $\leq n$, and that R is bounded with respect to the σ_i. By Lemma 3.7, there is a valuation ring $\mathcal{O}_i$ containing R and inducing σ_i; since the topologies are pairwise distinct, the $\mathcal{O}_i$ must be incomparable, and therefore $m \leq n$.

Theorem 4.10. *Let (K, τ) be a field with a W_n-topology.*
(1) There is at least one V-topological coarsening of τ.
(2) There are at most n such coarsenings.
(3) If (K, τ) is a definable topology (wrt to some structure on K), then every V-topological coarsening of τ is definable.

---

(3) Let σ be a V-topological coarsening of τ. Let D and B be bounded neighbourhoods of 0 for τ and σ respectively, with D definable. Wma D is a W_n-set, B is a W_1-set. Since $\sigma \subset \tau$, D is also σ-bounded. Replacing B by $B \cup D$, wma $D \subseteq B$. Consider an ultrapower (K^*, D^*, B^*), R_D and R_B the associated subrings generated by D^* and B^*. Then
R_D is a W_n-subring of K^*, $\vee$-definable in (K^*, D^*), and co-embeddable with D^*;
R_B is a W_1-subring of K^*, $\vee$-definable in (K^*, B^*), co-embedddable with B^*.

$R_B \supseteq R_D$. As R_B is a valuation ring, it contains $\tilde{R}_D$; But $\tilde{R}_D$ is an intersection of valuation rings $\mathcal{O}_1, \ldots, \mathcal{O}_n$, hence R_B must contain some $\mathcal{O}_i$.

So $\mathcal{O}_i$ is $\vee$-definable in (K^*, D^*). By Prop. 4.6, there is a definable $C \subset K^*$ which is co-embeddable with $\mathcal{O}_i$. Note that $R_B \supset \mathcal{O}_i$ implies they induce the same topology, and therefore $C, \mathcal{O}_i, R_B, B^*$ are pairwise co-embeddable.
Therefore there is a set C_0 definable in (K, D) which is co-embeddable with B (work in the structure (K, D, B)). So, σ is definable in (K, D).

Proposition 4.11. *Let* $(\mathbb{K}, \tau, \ldots)$ *be a sufficiently saturated field, and* τ *a definable* W_n*-topology on* $\mathbb{K}$*. Then* τ *is induced by a* $\vee$*-definable, externally definable* W_n*-ring* R *on* $\mathbb{K}$*.*

Proof. Let D be a definable bounded neighbourhood of 0 which is a W_n-set. Wma the language is countable, and we let $K \prec \mathbb{K}$, K countable, over which D is definable. Let R be the union of all K-definable bounded sets. Then R is a $\vee$-definable ring, which contains D, so is a W_n-ring, and is bounded. So R induces the topology (bounded neighbourhood).

Claim. If $S_1, \ldots, S_r$ are K-definable bounded subsets of $\mathbb{K}$, then there is $c \in K^\times$ such that $\bigcup S_i \subseteq cD$.
There is such a $c \in \mathbb{K}^\times$, because $\bigcup S_i$ is bounded. But since everything is K-definable, such a c exists in K.

So, $R = \bigcup_{c \in K^\times} cD$, a directed union. So R is externally definable: if $\varphi(y)$ defines D, consider the following partial type over $\mathbb{K}$:

$$\Sigma(x) = \{\forall y\, \varphi(c^{-1}y) \to \varphi(x^{-1}y) \mid c \in K\} \cup \{\exists y\, \varphi(c \in y) \wedge \neg\varphi(x^{-1}y) \mid cD \not\subseteq R\}.$$

If c^* realises Σ in some $\mathbb{K} \prec \mathbb{K}^*$, then $c^*\varphi(\mathbb{K}^*) \cap \mathbb{K} = R$.

Definition 1.5. Let K be a field. A *golden lattice* on K is a collection Λ of additive subgroups of K, which contains $\{0\}$ and K but also other elements, is closed under (finite) intersection, sum and scalar multiplication, has finite cube rank, and the set $\Lambda^+ = \Lambda \setminus \{0\}$ is closed under intersection.

Theorem 5.9. *If Λ is a golden lattice on K, then Λ^+ is a neighbourhood basis of a W-topology on K. If Λ has rank r, this is a W_r-topology.*

Proof. We check the relevant axioms, U, V, W will range over Λ^+. First, Λ^+ is a filter base, and is non-discrete. Let $U \in \Lambda$ with $0 < U < K$; if $0 \neq a \in K$, and $b \in K \setminus U$, then $a = (ab^{-1})b \notin (ab^{-1})U$. If $U \in \Lambda^+$, then $U - U = U$, so $+$ is continuous, as is multiplication by a scalar. For multiplication we need to work.

Step 1. Choose $V_1 \in \Lambda^+$ such that c−rk$(K/V_1) = r =$ c−rk(Λ). If $r = 1$, any element of Λ^+ which is $\neq K$ will do. Assume $r > 1$, and consider a strict r-cube in Λ^+; find $B_1, \ldots, B_r$ in this r-cube which are independent over the base $A = \bigcap B_i$ and note that $A \neq \{0\}$ by goldenness.

Step 2. If $a_i \in B_i \setminus A$, then $S = \{a_1, \ldots, a_r\}$ has the following property: whenever $B \in \Lambda$ contains S, then B contains A. (Lemma 5.7: show that the $(A + C) \cap B_i$ form a strict r-cube containing A, then that c$-$rk$(A/A \cap C) = 0$).

Step 3. If we set $V_2 = \bigcap a_i^{-1} U$ and $V = V_1 \cap V_2$, then $V \cdot V \subseteq U$. Indeed, if $a, b \in V$, then $a_i \in a^{-1}U$ and therefore $S \subseteq a^{-1}U \in \Lambda$, whence $V_1 \subseteq a^{-1}U$, and $b \in a^{-1}U$, i.e., $ab \in U$.

Locally bounded?

Let $U = V_1$ and $S = \{a_1, \ldots, a_r\}$ be as in claim 1. If $V \in \Lambda^+$, and $0 \neq c \in \bigcap a_i^{-1} V$, then $cS \subseteq V$, whence $cU \subseteq V$.

W_r?

Again let $U \in \Lambda^+$ be such that c$-$rk$(K/U) = r$, $1 \in U$. We know that U is a bounded neighbourhood of 0. If $a_1, \dots, a_{r+1} \in U$, then $\sum_i a_i U = \sum_{i \neq j} a_i U$ for some j, and therefore $a_j \in a_j U \subseteq \sum_{i \neq j} a_i U$.

Remark: *this characterizes bounded sets as those $A \in \Lambda^+$ with c$-$rk$(K/A) = r$.*

All results of §6 use the notion of infinitesimals, which we didn't do in detail. They appear in dpII section 3.

Lemma 7.1. *Let τ, τ' be two ring topologies on the field K, with $\tau' \subseteq \tau$. If τ is a W_n-topology, so is τ'.*

Proof. The proof is surprisingly long. The difficulty lies in showing that τ' is locally bounded. Given that there are local sentences expressing that τ is W_n, $\tau \subseteq \tau'$, and τ' is not W_n, we may assume that (K, τ, τ') is ω-complete, whence τ is induced by a W_n-subring R.

Claim 1. R is τ'-bounded.
If $U \in \tau' \subseteq \tau$, then for some $c \in K^\times$, $cR \subseteq U$.

Claim 2. if $U \in \tau'$, there is $M \in \tau'$, $M \subset U$, such that M is an R-submodule of K.
Choose a descending chain (U_i) of elements of τ', with $U_0 = U$, satisfying

$$U_{n+1} \cup (U_{n+1} - U_{n+1}) \cup U_{n+1}R \subset U_n.$$

This is possible because R is τ'-bounded, and τ' is a ring topology.
Then $M = \bigcap U_n$ belongs to τ', and is an R-submodule of K.

Claim 1. R is τ'-bounded.
Claim 2. if $U \in \tau'$, there is $M \in \tau'$, $M \subset U$, such that M is an R-submodule of K.

We now consider the set Λ^+ of all R-submodules of K which are in τ', and define $\Lambda = \Lambda^+ \cup \{0\}$. Then Λ is a golden lattice, and by Thm 5.9, Λ^+ defines a W_n-topology τ'' on K, and we have $\tau'' = \tau'$ by Claim 2.

Lemma 7.3. *Let $R \subseteq R'$ be two subrings of $K = Frac(R)$. If R is a W_n-ring, then either R' is a W_{n-1}-ring, or R and R' are co-embeddable.*

Proof. We know that the cube rank $\text{c}-\text{rk}_{R'}(K) \leq \text{c}-\text{rk}_R(K)$. So, assume both ranks are equal to n; we need to show that R' is embeddable in R. We look at the lattices Λ and Λ' of R-, resp. R'-submodules of K; they are golden lattices, with $\Lambda' \subseteq \Lambda$. Take $A \in \Lambda'$ such that A is the base of a strict n-cube in Λ'; then this is also a strict n-cube in Λ, and therefore $\text{c}-\text{rk}_R(K/A) = \text{c}-\text{rk}_{R'}(K/A) = \text{c}-\text{rk}_R(K) = \text{c}-\text{rk}_{R'}(K)$. By the remark after 5.10, A is bounded in both topologies.

Corollary 7.4. *If τ is a W_n-topology on a field K, and τ' is a strict coarsening of τ, then τ' is a W_{n-1}-topology.*

Independence and approximation

Definition 1.6. Let τ, τ' be two ring topologies on K, Then τ and τ' are *independent* iff every non-empty τ-open set intersects non-trivially every non-empty τ'-open set.

Remarks: This is a local sentence. It can also be expressed by: $\forall U \in \tau, \forall V \in \tau', \; U + V = K$.

Lemma 7.15. *Let R be a W_n-ring on K, and R_i, $i = 1, 2$, subrings of K containing R. Then either there is a V-topology coarser than both τ_{R_1} and τ_{R_2}, or τ_{R_1} and τ_{R_2} are independent.*

Proof. Let Λ^+ be the set of R-submodules M of K which are neighbourhoods of 0 in both τ_{R_1} and τ_{R_2}; if $\Lambda^+ \neq \{K\}$, then $\Lambda = \Lambda^+ \cup \{0\}$ is a golden lattice, of rank $\leq n$. Assume that τ_{R_1} and τ_{R_2} are not independent, and take non-zero ideals $I_1 \leq R_1$ and $I_2 \leq R_2$ such that $I_1 + I_2 < K$. Then $I_1 + I_2 \in \Lambda$, so that Λ is golden. Then Λ^+ defines a W_n-topology τ' on K, which is a common coarsening of τ_{R_1} and τ_{R_2}; and we apply 4.10: there is a V-topology which is coarser than τ', hence than τ_{R_1} and τ_{R_2}.

Corollary 7.16. *Let τ_0, τ_1, τ_2 be W-topologies on K, with τ_0 finer than τ_1 and τ_2. Then either τ_1 and τ_2 are independent, or they share a common V-topological coarsening.*

Proof. We may pass to a saturated extension, in which all topologies τ_i are coming from rings R_i, and we have $R_0 \subseteq R_1, R_2$. Now apply the previous result.