dp-finite fields VI, part 1

dp-finite fields reading group – MSRI

Blaise Boissonneau
PhD student of Franziska Jahnke

WWU, Münster

November 5, 2020

Goal

Theorem (4.3)

Let τ be a local W-topology on a field K, then τ has a unique V-topological coarsening.
Equivalently, in an ultrapower K^*, the integral closure $\widetilde{R}$ of R_τ is a valuation ring.

Multi-approximation

We work over a fixed infinite field K_0. We assume all valuations are trivial when restricted to K_0.

Multi-approximation

We work over a fixed infinite field K_0. We assume all valuations are trivial when restricted to K_0.

Lemma (3.1)

Let $v_1, \ldots, v_m$ be valuations on a field K. For any $z, w \in K$, there is $c \in K_0$ such that $v_i(z - cw) = \min(v_i(z), v_i(w))$ for all i.

Multi-approximation

We work over a fixed infinite field K_0. We assume all valuations are trivial when restricted to K_0.

Lemma (3.1)

Let $v_1, \ldots, v_m$ be valuations on a field K. For any $z, w \in K$, there is $c \in K_0$ such that $v_i(z - cw) = \min(v_i(z), v_i(w))$ for all i.

Proof.

Taking $c \in K_0$ such that $c \neq \mathrm{res}_i(\frac{z}{w})$ works. □

Scrambling

Definition

$\overline{x} = (x_1, \ldots, x_n)$ is scrambled if $v_i(x_j) = v_i(x_k)$ for all $i \leqslant m$ and $j, k \leqslant n$.

Scrambling

Definition

$\overline{x} = (x_1, \ldots, x_n)$ is scrambled if $v_i(x_j) = v_i(x_k)$ for all $i \leqslant m$ and $j, k \leqslant n$.

$\mu \in \mathrm{GL}_n(K)$ scrambles $\overline{x}$ if $\mu \cdot \overline{x}$ is scrambled.

Scrambling

Definition

$\overline{x} = (x_1, \ldots, x_n)$ is scrambled if $v_i(x_j) = v_i(x_k)$ for all $i \leqslant m$ and $j, k \leqslant n$.

$\mu \in \mathrm{GL}_n(K)$ scrambles $\overline{x}$ if $\mu \cdot \overline{x}$ is scrambled.

Lemma (3.2)

Let $v_1, \cdots, v_m$ be valuations on K. Any n-tuple $\overline{x}$ can be scrambled by some $\mu \in \mathrm{GL}_n(K_0)$.

Scrambling

Definition

$\overline{x} = (x_1, \ldots, x_n)$ is scrambled if $v_i(x_j) = v_i(x_k)$ for all $i \leqslant m$ and $j, k \leqslant n$.

$\mu \in \mathrm{GL}_n(K)$ scrambles $\overline{x}$ if $\mu \cdot \overline{x}$ is scrambled.

Lemma (3.2)

Let $v_1, \cdots, v_m$ be valuations on K. Any n-tuple $\overline{x}$ can be scrambled by some $\mu \in \mathrm{GL}_n(K_0)$.

Proof.

Whenever $v_i(x_j) > v_i(x_k)$, we find $c \in K_0$ such that $v_{i'}(x_j - cx_k) = \min(v_{i'}(x_j), v_{i'}(x_k))$ for all i', and we replace x_j by $x_j - cx_k$. Iterating this process gives a scrambled vector in finitely many steps. □

Scrambling

Definition

$\overline{x} = (x_1, \ldots, x_n)$ is scrambled if $v_i(x_j) = v_i(x_k)$ for all $i \leqslant m$ and $j, k \leqslant n$.
$\mu \in \mathrm{GL}_n(K)$ scrambles $\overline{x}$ if $\mu \cdot \overline{x}$ is scrambled.

Lemma (3.2)

Let $v_1, \cdots, v_m$ be valuations on K. Any n-tuple $\overline{x}$ can be scrambled by some $\mu \in \mathrm{GL}_n(K_0)$.

Proof.

Whenever $v_i(x_j) > v_i(x_k)$, we find $c \in K_0$ such that $v_{i'}(x_j - cx_k) = \min(v_{i'}(x_j), v_{i'}(x_k))$ for all i', and we replace x_j by $x_j - cx_k$. Iterating this process gives a scrambled vector in finitely many steps. □

Corollary (3.3)

For every n, m there is a finite $G_{n,m} \subset \mathrm{GL}_n(K_0)$ such that in any m-valued field, any n-tuple is scrambled by an element of $G_{n,m}$.

R-independence

Lemma (3.5)

Let R be a weight n local integral domain, let $R \subsetneq \mathrm{Frac}(R) = K$, let $\tau = \tau_R$ be local and weight n. Consider $\widetilde{R} = \mathcal{O}_1 \cap \cdots \cap \mathcal{O}_m$, the integral closure of R. Fix $a \in R \setminus \{0\}$. Then we can find $y_1, \ldots, y_n \in K$ scrambled and $(a^{-1}R)$-independent.

Note that we are still working over K_0.

R-independence

Lemma (3.5)

Let R be a weight n local integral domain, let $R \subsetneq \mathrm{Frac}(R) = K$, let $\tau = \tau_R$ be local and weight n. Consider $\widetilde{R} = \mathcal{O}_1 \cap \cdots \cap \mathcal{O}_m$, the integral closure of R. Fix $a \in R \setminus \{0\}$. Then we can find $y_1, \ldots, y_n \in K$ scrambled and $(a^{-1}R)$-independent.

Note that we are still working over K_0.

Proof.

Consider the following formula:

$$\varphi(\overline{x}) : \bigwedge_{\mu \in G_{n,m}} \text{“}\mu \cdot \overline{x} \text{ is } (a^{-1}R)\text{-independent”}.$$

R-independence

Lemma (3.5)

Let R be a weight n local integral domain, let $R \subsetneq \mathrm{Frac}(R) = K$, let $\tau = \tau_R$ be local and weight n. Consider $\widetilde{R} = \mathcal{O}_1 \cap \cdots \cap \mathcal{O}_m$, the integral closure of R. Fix $a \in R \setminus \{0\}$. Then we can find $y_1, \ldots, y_n \in K$ scrambled and $(a^{-1}R)$-independent.

Note that we are still working over K_0.

Proof.

Consider the following formula:

$$\varphi(\overline{x}) : \bigwedge_{\mu \in G_{n,m}} \text{“}\mu \cdot \overline{x} \text{ is } (a^{-1}R)\text{-independent”}.$$

In an ultrapower (K^*, R^*), this formula is realised; in fact any R_τ-independent n-tuple realises it.

R-independence

Lemma (3.5)

Let R be a weight n local integral domain, let $R \subsetneq \mathrm{Frac}(R) = K$, let $\tau = \tau_R$ be local and weight n. Consider $\widetilde{R} = \mathcal{O}_1 \cap \cdots \cap \mathcal{O}_m$, the integral closure of R. Fix $a \in R \setminus \{0\}$. Then we can find $y_1, \ldots, y_n \in K$ scrambled and $(a^{-1}R)$-independent.

Note that we are still working over K_0.

Proof.

Consider the following formula:

$$\varphi(\overline{x}) : \bigwedge_{\mu \in G_{n,m}} \text{“}\mu \cdot \overline{x} \text{ is } (a^{-1}R)\text{-independent”}.$$

In an ultrapower (K^*, R^*), this formula is realised; in fact any R_τ-independent n-tuple realises it.
Thus φ is realised in K, and there is $\mu \in G_{n,m}$ such that $\overline{y} = \mu \cdot \overline{x}$ is scrambled. □

Density of R

Lemma (3.7)

Keeping previous notations, R is dense in $J(\widetilde{R}) = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_m$ with respect to the topology $\widetilde{\tau} = \tau_{\widetilde{R}}$.

Density of R

Lemma (3.7)

Keeping previous notations, R is dense in $J(\widetilde{R}) = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_m$ with respect to the topology $\widetilde{\tau} = \tau_{\widetilde{R}}$.

Proof.

Let $x \in J(\widetilde{R})$. For any $U \in \widetilde{\tau}$, we want to find $b \in (x + U) \cap R$.

Density of R

Lemma (3.7)

Keeping previous notations, R is dense in $J(\widetilde{R}) = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_m$ with respect to the topology $\widetilde{\tau} = \tau_{\widetilde{R}}$.

Proof.

Let $x \in J(\widetilde{R})$. For any $U \in \widetilde{\tau}$, we want to find $b \in (x + U) \cap R$.

Take $a \in R \cap x^{-1}R \cap J(\widetilde{R})$ non-zero and such that $a\widetilde{R} \subset U$. We can find $\overline{y} \in K^n$ scrambled and $(a^{-2}R)$-independent. Scaling it, we may assume $y_1 = 1$ and thus $\overline{y} \in \widetilde{R}^n$.

Density of R

Lemma (3.7)

Keeping previous notations, R is dense in $J(\widetilde{R}) = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_m$ with respect to the topology $\widetilde{\tau} = \tau_{\widetilde{R}}$.

Proof.

Let $x \in J(\widetilde{R})$. For any $U \in \widetilde{\tau}$, we want to find $b \in (x + U) \cap R$.

Take $a \in R \cap x^{-1}R \cap J(\widetilde{R})$ non-zero and such that $a\widetilde{R} \subset U$. We can find $\overline{y} \in K^n$ scrambled and $(a^{-2}R)$-independent. Scaling it, we may assume $y_1 = 1$ and thus $\overline{y} \in \widetilde{R}^n$.

As R is of weigth exactly n, the set $\{1, x, ay_2, \ldots, ay_n\}$ is not R-independent. The only possibility is to have $x \in R + aRy_2 + \cdots + aRy_n$, which means $x = b + ar_2y_2 + \cdots + ar_ny_n$; thus $x - b \in \widetilde{R} \subset U$. □

Moving valuation rings

Lemma (3.8)

Keeping previous notations, we consider $\widetilde{R} = \mathcal{O}_1 \cap \cdots \cap \mathcal{O}_m$*; suppose the* $\mathcal{O}_i$ *are pairwise independent. Then there is* $a \in K^\times$ *such that for every* $b \in K^\times$*,* $R \cap b\mathcal{O}_2 \not\subseteq a\mathcal{O}_1$*.*

Moving valuation rings

Lemma (3.8)

Keeping previous notations, we consider $\widetilde{R} = \mathcal{O}_1 \cap \cdots \cap \mathcal{O}_m$; suppose the $\mathcal{O}_i$ are pairwise independent. Then there is $a \in K^\times$ such that for every $b \in K^\times$, $R \cap b\mathcal{O}_2 \not\subseteq a\mathcal{O}_1$.

Proof.

Take a non-zero $c \in \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_m$. By the approximation theorem, we can find $a \in K$ such that $v_1(a) > v_1(c)$ and $v_i(a) = v_i(c)$ for $i \neq 1$.

Moving valuation rings

Lemma (3.8)

Keeping previous notations, we consider $\widetilde{R} = \mathcal{O}_1 \cap \cdots \cap \mathcal{O}_m$; suppose the $\mathcal{O}_i$ are pairwise independent. Then there is $a \in K^\times$ such that for every $b \in K^\times$, $R \cap b\mathcal{O}_2 \not\subseteq a\mathcal{O}_1$.

Proof.

Take a non-zero $c \in \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_m$. By the approximation theorem, we can find $a \in K$ such that $v_1(a) > v_1(c)$ and $v_i(a) = v_i(c)$ for $i \neq 1$.

Fix $b \in K^\times$. Again, we can find $u \in K$ such that $v_2(u) > \max(v_2(b), v_2(c))$ and $v_i(u) = v_i(c)$ for $i \neq 2$.

Moving valuation rings

Lemma (3.8)

Keeping previous notations, we consider $\widetilde{R} = \mathcal{O}_1 \cap \cdots \cap \mathcal{O}_m$; suppose the $\mathcal{O}_i$ are pairwise independent. Then there is $a \in K^\times$ such that for every $b \in K^\times$, $R \cap b\mathcal{O}_2 \not\subseteq a\mathcal{O}_1$.

Proof.

Take a non-zero $c \in \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_m$. By the approximation theorem, we can find $a \in K$ such that $v_1(a) > v_1(c)$ and $v_i(a) = v_i(c)$ for $i \neq 1$.

Fix $b \in K^\times$. Again, we can find $u \in K$ such that $v_2(u) > \max(v_2(b), v_2(c))$ and $v_i(u) = v_i(c)$ for $i \neq 2$.

Now $u \in b\mathcal{O}_2$ and not in $a\mathcal{O}_1$. Since $u \in \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_m$, we conclude by density of R. □

V-topological coarsenings

We now consider arbitrary fields – not necessarily extending a trivially valued infinite K_0.

Lemma (4.1)

Let τ be a local weight n topology on an infinite field K. Suppose τ_1 and τ_2 are 2 different V-topological coarsening of τ. Then for all $U \in \tau$ there is $V \in \tau_1$ such that for all $W \in \tau_2$, we have $U \cap W \not\subseteq V$.

V-topological coarsenings

We now consider arbitrary fields – not necessarily extending a trivially valued infinite K_0.

Lemma (4.1)

Let τ be a local weight n topology on an infinite field K. Suppose τ_1 and τ_2 are 2 different V-topological coarsening of τ. Then for all $U \in \tau$ there is $V \in \tau_1$ such that for all $W \in \tau_2$, we have $U \cap W \nsubseteq V$.

Proof.

We move to an ultrapower $(K^*, \tau^*, \tau_1^*, \tau_2^*)$ and consider R, R_1 and R_2, the rings associated with τ, τ_1 and τ_2.

V-topological coarsenings

We now consider arbitrary fields – not necessarily extending a trivially valued infinite K_0.

Lemma (4.1)

Let τ be a local weight n topology on an infinite field K. Suppose τ_1 and τ_2 are 2 different V-topological coarsening of τ. Then for all $U \in \tau$ there is $V \in \tau_1$ such that for all $W \in \tau_2$, we have $U \cap W \nsubseteq V$.

Proof.

We move to an ultrapower $(K^*, \tau^*, \tau_1^*, \tau_2^*)$ and consider R, R_1 and R_2, the rings associated with τ, τ_1 and τ_2.

We write $\widetilde{R} = R_1 \cap R_2 \cap \mathcal{O}_3 \cap \cdots \cap \mathcal{O}_m$. We want to show: $\forall c \neq 0 \exists a \neq 0 \forall b \neq 0,\ cR \cap bR_2 \nsubseteq aR_1$.

V-topological coarsenings

We now consider arbitrary fields – not necessarily extending a trivially valued infinite K_0.

Lemma (4.1)

Let τ be a local weight n topology on an infinite field K. Suppose τ_1 and τ_2 are 2 different V-topological coarsening of τ. Then for all $U \in \tau$ there is $V \in \tau_1$ such that for all $W \in \tau_2$, we have $U \cap W \nsubseteq V$.

Proof.

We move to an ultrapower $(K^*, \tau^*, \tau_1^*, \tau_2^*)$ and consider R, R_1 and R_2, the rings associated with τ, τ_1 and τ_2.

We write $\widetilde{R} = R_1 \cap R_2 \cap \mathcal{O}_3 \cap \cdots \cap \mathcal{O}_m$. We want to show: $\forall c \neq 0 \exists a \neq 0 \forall b \neq 0,\ cR \cap bR_2 \nsubseteq aR_1$.

Dividing by c this is the conclusion of the previous lemma – Applied on K^* above the field $K_0 = K$.

□

Comparing max ideals

Lemma (4.2)

Let τ be a local weight n topology on an infinite field K, let $R = R_\tau \subseteq K^*$, and let $\widetilde{R} = \mathcal{O}_1 \cap \cdots \cap \mathcal{O}_m$ be its integral closure. Then $\mathfrak{m}_i \cap R \not\subseteq \mathfrak{m}_j \cap R$ for any $i \neq j$.

Comparing max ideals

Lemma (4.2)

Let τ be a local weight n topology on an infinite field K, let $R = R_\tau \subseteq K^*$, and let $\widetilde{R} = \mathcal{O}_1 \cap \cdots \cap \mathcal{O}_m$ be its integral closure. Then $\mathfrak{m}_i \cap R \not\subseteq \mathfrak{m}_j \cap R$ for any $i \neq j$.

Proof.

Let τ_i, τ_j be the topology corresponding to $\mathcal{O}_i, \mathcal{O}_j$. Fix $U \in \tau$ bounded. Let $V \in \tau_j$ be such that for all $W \in \tau_i$, $U \cap W \not\subseteq V$.

Comparing max ideals

Lemma (4.2)

Let τ be a local weight n topology on an infinite field K, let $R = R_\tau \subseteq K^*$, and let $\widetilde{R} = \mathcal{O}_1 \cap \cdots \cap \mathcal{O}_m$ be its integral closure. Then $\mathfrak{m}_i \cap R \nsubseteq \mathfrak{m}_j \cap R$ for any $i \neq j$.

Proof.

Let τ_i, τ_j be the topology corresponding to $\mathcal{O}_i, \mathcal{O}_j$. Fix $U \in \tau$ bounded. Let $V \in \tau_j$ be such that for all $W \in \tau_i$, $U \cap W \nsubseteq V$.

Take $\varepsilon \in U^* \cap \bigcap_{W \in \tau_i} W^*$ but outside of V^*. We have $\varepsilon \in \mathfrak{m}_i$ but $\varepsilon \notin \mathfrak{m}_j$.

Comparing max ideals

Lemma (4.2)

Let τ be a local weight n topology on an infinite field K, let $R = R_\tau \subseteq K^*$, and let $\widetilde{R} = \mathcal{O}_1 \cap \cdots \cap \mathcal{O}_m$ be its integral closure. Then $\mathfrak{m}_i \cap R \not\subseteq \mathfrak{m}_j \cap R$ for any $i \neq j$.

Proof.

Let τ_i, τ_j be the topology corresponding to $\mathcal{O}_i, \mathcal{O}_j$. Fix $U \in \tau$ bounded. Let $V \in \tau_j$ be such that for all $W \in \tau_i$, $U \cap W \not\subseteq V$.

Take $\varepsilon \in U^* \cap \bigcap_{W \in \tau_i} W^*$ but outside of V^*. We have $\varepsilon \in \mathfrak{m}_i$ but $\varepsilon \notin \mathfrak{m}_j$.

Finally $\varepsilon \in R$ because $\varepsilon \in U^*$ and U is bounded. □

Main result

Theorem (4.3)

Let τ be a local W-topology on a field K, then τ has a unique V-topological coarsening.
Equivalently, in an ultrapower K^*, the integral closure $\widetilde{R}$ of R_τ is a valuation ring.

Main result

Theorem (4.3)

Let τ be a local W-topology on a field K, then τ has a unique V-topological coarsening.
Equivalently, in an ultrapower K^*, the integral closure $\widetilde{R}$ of R_τ is a valuation ring.

Proof.

We write $\widetilde{R} = \mathcal{O}_1 \cap \ldots \mathcal{O}_m$. For each i, $\mathfrak{p}_i = \mathfrak{m}_i \cap R$ is a prime ideal of R. By the previous lemma, the $\mathfrak{p}_i$ are pairwise incomparable.

Main result

Theorem (4.3)

Let τ be a local W-topology on a field K, then τ has a unique V-topological coarsening.
Equivalently, in an ultrapower K^*, the integral closure $\widetilde{R}$ of R_τ is a valuation ring.

Proof.

We write $\widetilde{R} = \mathcal{O}_1 \cap \ldots \mathcal{O}_m$. For each i, $\mathfrak{p}_i = \mathfrak{m}_i \cap R$ is a prime ideal of R. By the previous lemma, the $\mathfrak{p}_i$ are pairwise incomparable.

Let $\mathfrak{p}$ be the max ideal of R. Let $\mathcal{O}$ be a valuation ring such that $R \subseteq \mathcal{O}$ and $\mathfrak{m} \cap R = \mathfrak{p}$.

Main result

Theorem (4.3)

Let τ be a local W-topology on a field K, then τ has a unique V-topological coarsening.
Equivalently, in an ultrapower K^*, the integral closure $\widetilde{R}$ of R_τ is a valuation ring.

Proof.

We write $\widetilde{R} = \mathcal{O}_1 \cap \ldots \mathcal{O}_m$. For each i, $\mathfrak{p}_i = \mathfrak{m}_i \cap R$ is a prime ideal of R. By the previous lemma, the $\mathfrak{p}_i$ are pairwise incomparable.

Let $\mathfrak{p}$ be the max ideal of R. Let $\mathcal{O}$ be a valuation ring such that $R \subseteq \mathcal{O}$ and $\mathfrak{m} \cap R = \mathfrak{p}$.

We have $\widetilde{R} \subseteq \mathcal{O}$ so there is i such that $\mathcal{O}_i \subseteq \mathcal{O}$.

Main result

Theorem (4.3)

Let τ be a local W-topology on a field K, then τ has a unique V-topological coarsening.
Equivalently, in an ultrapower K^*, the integral closure $\widetilde{R}$ of R_τ is a valuation ring.

Proof.

We write $\widetilde{R} = \mathcal{O}_1 \cap \ldots \mathcal{O}_m$. For each i, $\mathfrak{p}_i = \mathfrak{m}_i \cap R$ is a prime ideal of R. By the previous lemma, the $\mathfrak{p}_i$ are pairwise incomparable.

Let $\mathfrak{p}$ be the max ideal of R. Let $\mathcal{O}$ be a valuation ring such that $R \subseteq \mathcal{O}$ and $\mathfrak{m} \cap R = \mathfrak{p}$.

We have $\widetilde{R} \subseteq \mathcal{O}$ so there is i such that $\mathcal{O}_i \subseteq \mathcal{O}$.

This implies $\mathfrak{p} \subseteq \mathfrak{p}_i$, and since $\mathfrak{p}_j \subseteq \mathfrak{p}$, they can't be pairwise incomparable unless there's only 1. □

Questions? Remarks?

Questions? Remarks?

Questions? Remarks?

Questions? Remarks?