Dp-finite fields reading seminar
paper VI, §4

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Motivation

To prove:

Theorem (Theorem 1.10)

Let (K, τ) be a W-topological field. Then there exist W-topological coarsenings $\tau_1, \ldots, \tau_n$ such that

1. *the τ_i are jointly independent and generate τ,*
2. *each τ_i has a unique V-topological coarsening, giving a bijection between $\{\tau_1, \ldots, \tau_n\}$ and the V-topological coarsenings of τ.*

This will follow from theorems 4.10 and 4.13.

Motivation

Proposition 2.5 (Dictionary)

Let $R \subseteq K^*$ be $\bigvee$-definable over K. Suppose that $K \subseteq R \subset \mathrm{Frac}(R) = K^*$.

1. $R = R_\tau$ for some topology τ on K iff R is co-embeddable with a definable set.
2. $R = R_\tau$ for some W_n-topology τ iff R is a W_n-ring.
3. $R = R_\tau$ for some V-topology τ iff R is a valuation ring.

Proposition 2.12

Let $R \subseteq K^*$ be Let τ be a W-topology with corresponding W-ring $R_\tau \subseteq K^*$ that is $\bigvee$-definable over K.

4. τ is local iff R_τ is a local ring.

Definition

Let τ be a W-topology, with corresponding ring $R_\tau \subseteq K^*$.

- The *integral closure* is the (unique) W-topology corresponding to the integral closure of R_τ.
- The *local components* are the W-topologies corresponding to localisations of R at a maximal ideal.

Lemma (Corollary 6.7, paper II)

If R is W_n-ring, then $\widetilde{R}$ is an intersection of finitely many valuation rings, which may be chosen to be its localisations at its maximal ideals.

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Proposition 2.10

Let τ be a W-topology. Then the V-topological coarsenings of τ are exactly the local components of its integral closure.

Recap: section 4

Lemma 4.1

Let (K, τ) be a local topological field of weight n. Let τ_1, τ_2 be two distinct V-topological coarsesnings of τ. Then for all $U \in \tau$ there is $V \in \tau_1$ such that for all $W \in \tau_2$,

$$U \cap W \not\subseteq V.$$

Lemma 4.2

Let (K, τ) be a local topological field of weight n. Let K^* be an ultrapower of K. Let R be the subring of $K*$ induced by τ. Let $\widetilde{R}$ be the integral closure of R, and let

$$\widetilde{R} = \mathcal{O}_1 \cap \ldots \cap \mathcal{O}_n$$

be its decomposition into incomparable valuation rings. Let $\mathfrak{m}_i$ be the maximal ideal of $\mathcal{O}_i$. Then

$$\mathfrak{m}_i \cap R \not\subseteq \mathfrak{m}_j \cap R,$$

for $i \neq j$.

Theorem (4.3)

Let (K, τ) be a local W-topological field.

1. *τ has a unique V-topological coarsening.*
2. *If K^* is an ultrapower, if $R \subseteq K^*$ is the ring induced by τ, and $\widetilde{R}$ is its integral closure, then $\widetilde{R}$ is a valuation ring.*

‘Some commutative algebra’

Proposition 4.5

Let R be a domain.

1. R is the intersection of its key localisations.
2. If $\mathfrak{p}_1, \mathfrak{p}_2$ are two distinct maximal ideals of R, then $R_{\mathfrak{p}_1}$ is incomparable to $R_{\mathfrak{p}_2}$.
3. If $A \subseteq K$ is a local ring containing R, then A contains a key localisation of R.
4. The key localisations are the minimal local subrings of K containing R.

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Proof

For **1.**, let $R' = \bigcap_{\mathfrak{p} \in \mathrm{MaxSpec}(R)} R_{\mathfrak{p}}$. Then $R \subseteq R'$. Suppose $x \notin R$. Let $I = \{y \in R \mid xy \in R\} = R \cap \frac{1}{x}R$, which is a proper ideal in R (i.e. $1 \notin I$). Let $\mathfrak{p} \in \mathrm{MaxSpec}(R)$ s.t. $I \subseteq \mathfrak{p}$. Claim: $x \notin R_{\mathfrak{p}}$. Proof of claim: otherwise, $x = \frac{a}{s}$, some $a \in R$ and $s \in R \setminus \mathfrak{p}$. Then $s \in I$, contradiction to $I \subseteq \mathfrak{p}$. So $x \notin R_{\mathfrak{p}}$. $\square_{\text{claim}}$.

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For **2.**, suppose $\mathfrak{p}_1, \mathfrak{p}_2$ are distinct. Then $\mathfrak{p}_1, \mathfrak{p}_2$ are themselves incomparable. Let $x \in \mathfrak{p}_1 \setminus \mathfrak{p}_2$. Then $\frac{1}{x} \in R_{\mathfrak{p}_2}$. If also $\frac{1}{x} \in R_{\mathfrak{p}_1}$, then $\frac{1}{x} = \frac{a}{s}$ for $a \in R$ and $s \in R \setminus \mathrm{p}_1$. Then $s = ax \in \mathfrak{p}_1$, contradiction again! So $R_{\mathfrak{p}_2} \not\subseteq R_{\mathfrak{p}_1}$. By symmetry: $R_{\mathfrak{p}_1} \not\subseteq R_{\mathfrak{p}_2}$.

Proof cont.

For 3. and 4., let $\mathfrak{m}$ be the maximal ideal of A. Then $\mathfrak{m} \cap R$ is a proper prime ideal of R, therefore contained in some $\mathfrak{p} \in \mathrm{MaxSpec}(R)$. Obviously $R \subseteq A$. Let $x \in R \setminus \mathfrak{p}$. Then $x \notin \mathfrak{m}$, so $x \in A \setminus \mathfrak{m}$. Thus $x^{-1} \in A$. Therefore $R_{\mathfrak{p}} \subseteq A$.

Proposition 4.6

Let R be a domain. Let $R_1, \dots, R_n$ be among the key localisations of R. Let $R' = \bigcap_i R_i$. Then the key localisations of R' are $R_1, \dots, R_n$.

Proposition 4.7

Let R be a local domain with maximal ideal $\mathfrak{p}$. If the integral closure $\widetilde{R}$ is a local ring with maximal ideal $\mathfrak{q}$, then $\mathfrak{q} \cap R = \mathfrak{p}$.

Proposition 4.8

Let R be a domain with finitely many maximal ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$. Suppose that for each $R_{\mathfrak{p}_i}$, $\widetilde{R_{\mathfrak{p}_i}}$ is a valuation ring. Then

1. $\widetilde{R_{\mathfrak{p}_i}}$ is a key localisation of $\widetilde{R}$.
2. This gives a bijection between key localisations of R and key localisations of $\widetilde{R}$.

Independent topologies

Theorem 4.10

Let τ be a W-topology on K. Let $\tau_1, \ldots, \tau_n$ be the local components of τ.

1. Each τ_i has a unique V-topological coarsening.
2. This gives a bijection between the local components of τ and the V-topological coarsenings of τ.

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Proof.

1. is Theorem 4.3. For 2., take ultrapower K^*, let R, R_i be the $\bigvee$-definable rings corresponding to τ, τ_i. Simply by definition, the R_i are all the key localizations of R. By Theorem 4.3, each $\widetilde{R_i}$ is a valuation ring, corresponding to the unique V-topological coarsening of τ_i. By Lemma 4.8, the $\widetilde{R_i}$ are pairwise distinct, and are exactly the key localizations of $\widetilde{R}$. By Proposition 2.10, the key localizations of $\widetilde{R}$ correspond to the V-topological coarsenings of τ. □

Digression: independent sums of topologies

We can think about the following definition outside the context of fields.

Definition

Let T, T_i (for $i \in I$) be topologies on a set X. We say that T is an *independent sum* of the T_i if the diagonal map

$$(X, T) \longrightarrow \prod_i (X, T_i)$$

is an embedding with dense image.

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If $I = \{1, \ldots, n\}$ is finite, then the product has the *box topology* with basis

$$\{U_1 \times \ldots \times U_n \mid U_i \in T_i\}.$$

The topology induced on X by the diagonal map therefore has basis

$$\{U_1 \cap \ldots \cap U_n \mid U_i \in T_i\}.$$

For the image to be dense, it means that

$$U_1 \cap \ldots \cap U_n \neq \emptyset,$$

for $U_i \in T_i \setminus \{\emptyset\}$.

Putting this into the context of field topologies, and swapping topologies for filters, we restate the definition.

Definition (restated)

Let $\tau, \tau_1, \ldots, \tau_n$ be topologies on K. Say that τ is *independent sum* of the τ_i if the diagonal map

$$(K, \tau) \longrightarrow (K, \tau_1) \times \ldots \times (K, \tau_n)$$

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is an embedding with dense image.

Equivalently, if

$$\{U_1 \cap \ldots \cap U_n \mid U_i \in \tau_i\}$$

is a basis for τ and

$$(a_1 + U_1) \cap \ldots \cap (a_n + U_n) \neq \emptyset,$$

for all $a_i \in K$ and $U_i \in \tau_i$.

Lemma 4.12

If σ is an independent sum of $\tau_1, \ldots, \tau_{n-1}$ and τ is an independent sum of σ and τ_n, then τ is an independent sum of $\tau_1, \ldots, \tau_n$.

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Proof.

The composition of two (topological) embeddings with dense images is again a embedding with dense image. If $f : X \longrightarrow Y$ is such a map, and Z is any space, then

$$\begin{aligned} f \times \mathrm{id}_Z : X \times Z &\longrightarrow Y \times Z \\ (x, z) &\longmapsto (f(x), z) \end{aligned}$$

is again an embedding with dense image. The map we care about, namely

$$(K, \sigma) \longrightarrow (K, \tau_1) \times \ldots \times (K, \tau_n),$$

is therefore an embedding with dense image. □

Theorem (Theorem 4.13)

Let τ be a W-topology on a field K. Let $\tau_1, \ldots, \tau_n$ be the local components of τ. Then τ is an independent sum of the τ_i.

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Proof of 4.13

Fix an ultrapower K^* of K. Let $R, R_1, \ldots, R_n$ be the $\bigvee$-definable rings corresponding to $\tau, \tau_1, \ldots, \tau_n$. By the dictionary, the R_i are key localizations of R. For each i, let $S_i = R_1 \cap \ldots \cap R_i$. Note:

- $S_1 = R_1$ and $S_n = R$ (by Proposition 4.5),
- each S_i is a K-subalgebra of K^* which is $\bigvee$-definable over K, and each S_i is a W_n ring since $S_i \supseteq R$ (by Lemma 2.7 in paper V).

By dictionary, S_i corresponds to a W-topology σ_i. Note

- $\sigma_1 = \tau_1$ and $\sigma_n = \tau$.

The key localizations of S_i are $R_1, \ldots, R_i$, so the local components of σ_i are $\tau_1, \ldots, \tau_i$.
Consider the following claim....

Proof of 4.13

Claim

σ_i is an independent sum of σ_{i-1} and τ_i.

Proof of claim

First, to show independence of σ_{i-1} and τ_i. Let τ_j^V denote the unique V-topological coarsening of τ_j. By Theorem 4.10, the set of V-topological coarsenings of σ_{i-1} is

$$\{\tau_1^V, \ldots, \tau_{i-1}^V\}.$$

Note that τ_i^V is not in this set! So σ_{i-1} and τ_i have no common V-topological coarsening. Since also they are both coarsenings of τ which is a W-topology, it follows that they are independent (by Theorem 7.16, paper V).

Proof of 4.13

Proof of claim

Second, to show that σ_{i-1} and τ_i generate σ_i. Let's work in K^*, where $\sigma^*_{i-1}, \tau^*_i, \sigma^*_i$ denote the topologies corresponding to S_{i-1}, R_i, S_i. By definition $S_i = S_{i-1} \cap R_i$. For any nonzero a there are nonzero b, c such that

$$bS_{i-1} \cap cR_i \subseteq aS_i.$$

For $a = b = c$, equality! This proves

$$\forall U \in \sigma^*_i \exists V \in \sigma^*_{i-1} \exists W \in \tau^*_i \ : \ V \cap W \subseteq U.$$

Conversely,

$$\forall V \in \sigma^*_{i-1} \forall W \in \tau^*_i \exists U \in \sigma^*_i \ : \ U \subseteq V \cap W,$$

simply because we can take $U = V \cap W$. (Note $V, W \in \sigma^*_i$ since σ^*_i is finer than σ^*_{i-1} and τ^*_i.)
By properties of local sentences, both the above hold for $\sigma_{i-1}, \tau_i, \sigma_i$ in placed of their ultra-counterparts. Therefore σ_i is generated by σ_{i-1} and τ_i as required. $\square_{\text{claim}}$