Dp-finite fields V, section 6, (after Will Johnson)

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The aim of this talk is to understand the statement and ingredients of the following result:

Proposition 6.1. *Let T be a complete, dp-finite, unstable theory of fields, let $\mathbb{K}$ be a highly saturated model of T. Then*

• - *There is a small field $K \preceq \mathbb{K}$ such that the group J_K of infinitesimals is co-embeddable with a definable set D.*

• - *The canonical topology on $\mathbb{K}$ is a definable W_n-topology.*

We will also discuss the final step of the proof:

Theorem 6.8. (dpI) *Shelah's conjecture for dp-finite fields is implied by the following conjecture:*

If (K, τ) is a W-topological field of characteristic 0, then τ is generated by jointly independent topologies $\tau_1, \ldots, \tau_n$, and each τ_i has a unique V-topological coarsening.

We are getting close to the end of the proof. These two results are of course crucial in the proof of Shelah's conjecture for dp-finite fields. They rely however on several important results which appear in dpI and dpII, and haven't been done yet. I am thinking of:

dpI: 4.3, 4.18 to 4.22; 6.5, 6.16, 6.17, 6.19, 8.4, 10.1 (is not listed as purple, but should – absolutely crucial).

dpII: 5.9.2.

Heavy sets

The definition of heavy sets is . . . heavy: it involves the definition of *broad*, *narrow*, *quasi-minimal*, *coordinate configuration*, *critical rank*, and *critical set*. Fortunately there is an easier characterization, which we will also give.

We work in a monster model $\mathbb{K}$ of T, a dp-finite field with extra structure.

Definition. A definable set $X \subseteq \mathbb{K}$ is *heavy* if for some/any (dpI, 4.18, 4.19) critical set Y, there is $\delta \in \mathbb{K}$ such that dp$-$rk$(Y \cap X + \delta) =$ dp$-$rk(Y). Equivalently (dpII, 5.9.2), if dp$-$rk$(X) =$ dp$-$rk$(\mathbb{K})$. If it is not heavy then it is *light*.

Properties of heavy sets

Theorem 4.20 (dpI). *Let $X, Y \subseteq \mathbb{K}$ be definable. We suppose $\mathbb{K}$ infinite.*

The set of heavy subsets of $\mathbb{K}$ does not contain finite sets, contains $\mathbb{K}$, is closed under additive translation and multiplication by a non-zero element. A definable subset of a light set is light, and the union of two light sets is light. Further we have:

- *- If D_b is a definable family of subsets of $\mathbb{K}$, then $\{b \mid D_b$ is light $\}$ and $\{b \mid D_b$ is heavy $\}$ are definable (4.3(1), dpI).*
- *- If X and Y are heavy, then the set*

$$X -_{\infty} Y = \{\delta \in \mathbb{K} \mid X \cap (Y + \delta) \text{ is heavy}\}$$

is heavy.

This Theorem is proved before having the equivalent formulation.

Properties of heavy sets - 2

Lemmas 4.21, 4.22 (dpI). *Let* $M \prec \mathbb{K}$ *be a small model, over which some critical configuration is definable. Let* Z *be* M*-definable and heavy. Let* $D_1, \ldots, D_m$ *be* $\mathbb{K}$*-definable and such that*

$$Z(M) \subseteq D_1 \cup \cdots \cup D_m.$$

Then there is some j*, and an* M*-definable heavy set* $Z' \subseteq Z$ *such that* $Z'(M) \subseteq D_j$*.*

If W *is* $\mathbb{K}$*-definable and contains* $Z(M)$*, then* W *is heavy.*

Basic neighbourhoods, canonical topology

Definition. A *basic neighbourhood* is a set of the form $X -_\infty X$, where X is heavy.

Proposition 6.5 (dpI)

Let U, V be basic neighbourhoods. Then U is heavy, contains 0*; if $\alpha \neq$ 0, then αU is a basic neighbourhood; $U \cap V$ contains a basic neighbourhood W, which can be chosen M-definable if U and V are M-definable. If $\mathbb{K}$ is not of finite Morley rank, then for every $\alpha \neq$* 0*, there is a basic neighbourhood not containing α.*

(6.16 in dpI) *If U is a M-definable basic neighbourhood, then there is an M-definable basic neighbourhood V such that $V - V \subseteq U$.*

The *canonical topology* on $\mathbb{K}$ is the topology with basis of neighbourhoods of 0 the basic neighbourhoods, also called the *canonical basic neighbourhoods*.

Infinitesimals. $M \prec \mathbb{K}$.

Definition. $\varepsilon \in \mathbb{K}$ is an M-infinitesimal iff ε lies in every M-definable basic neighbouhood. Equivalently, if whenever X is heavy and M-definable, then $X \cap (X + \varepsilon)$ is heavy. The set of M-infinitesimals is denoted I_M (in dpI) or J_M (in dpV).

Theorem 6.17. (dpI) *If M is any model, then J_M is a subgroup of $(\mathbb{K}, +)$.*

Theorem 8.4. (dpI) *Let G be an abelian group, maybe with aditional structure, of dp-rank n. There is a cardinal κ such that for every type-definable $H \leq G$, $[H : H^{00}] < \kappa$. The same κ works for every $G \prec G^*$.*

Proposition 6.1. *Let T be a complete, dp-finite, unstable theory of fields, let $\mathbb{K}$ be a highly saturated model of T. Then*

- *- There is a small field $K \preceq \mathbb{K}$ such that the group J_K of infinitesimals is co-embeddable with a definable set D.*
- *- The canonical topology on $\mathbb{K}$ is a definable W_n-topology.*

Sketch of proof. Let $k_0 \prec \mathbb{K}$ be a subfield of cardinality κ, so that every type-definable k_0-linear subspace G of $\mathbb{K}$ has $G = G^{00}$. Let Λ be the lattice of type-definable k_0-linear subspaces of $\mathbb{K}$. Then this lattice is a golden lattice (10.1, dpI; needs infinite Morley rank): J_K belongs to Λ whenever $k_0 \preceq K \prec \mathbb{K}$; if $J \in \Lambda^+ = \Lambda \setminus \{0\}$, then every definable set containing J is heavy; moreover, if J is definable over K, then it contains J_K (scaling J, wma $1 \in J$, so that $k_0 \subseteq J$; by 4.22-dpI, J is heavy; the moreover clause follows from 6.19-dpI; note that it implies the same result for J type-definable); let $J_1, J_2 \in \Lambda^+$, choose $k_0 \prec K \prec \mathbb{K}$ over which they are type-defined; then $J_1 \cap J_2 \supseteq J_K \in \Lambda^+$. By earlier results (4.1) Λ^+ is a basis of neighbourhoods for some W_r-topology τ on $\mathbb{K}$. If $V \in \Lambda^+$ is bounded and ∞-definable over $K \supset k_0$, then (6.19 in dpI) $J_K \in \Lambda^+$ and $J_K \subseteq V$.

Then 4.1 gives us that the bounded ∞-K-definable group J_K is co-embeddable with some K-definable D, and the W_r-topology τ is defined by D. We need to show that τ is the canonical topology τ_{can}. Wma

$$J_K \subseteq D \subseteq eJ_K$$

for some $e \in K^\times$. Since J_K is a filtered intersection of K-definable canonical neighbourhoods, wma D is a K-definable canonical basic neighbourhood, i.e., $D \in \tau_{can}$. Moreover, if U is a K-definable canonical basic neighbourhood, then $e^{-1}D \subseteq U$. As $K \preceq \mathbb{K}$, the same is true of $\mathbb{K}$: if U is any $\mathbb{K}$-definable canonical basic neighbourhood, then there is $c \in \mathbb{K}^\times$ such that $cD \subseteq U$. So $\tau = \tau_{can}$.

Corollary 6.5. *The henselianity conjecture (that every NIP valued field is henselian) implies Shelah's conjecture for dp-finite fields.*

Let K be a dp-finite field. Wma K is unstable (Halevi-Palacin), and we know that for $K^* \succ K$, K^* has a definable W_n-topology, has a definable coarsening which is a V-topology, and is induced by some valuation subring of K^*; this valuation must be henselian, and therefore also p-henselian. We want to show that either some non-trivial henselian valuation is definable on K, or K is RCF.

If K is not RCF, neither is any finite algebraic extension of K, and we may therefore replace K by a finite algebraic extension L: it will have finite dp-rank, and wma that for some prime p, it contains a primitive p-th root of 1 (or $\sqrt{-1}$ if $p = 2 \neq \text{char}(K)$) and has a finite Galois extension of degree p. By a result of Jahnke-Koenigsmann, the field $L^* := K^*L$ admits a $\emptyset$-definable non-trivial p-Henselian valuation v, which must therefore be henselian. As L is K-definable in K and $L \prec K^*L$, the restriction of v to K is K-definable, and is non-trivial henselian.

Theorem 6.6. *Let K be an unstable dp-finite fields. The definable V-topologies on K are exactly the V-topological coarsenings of the canonical topology.*

That every V-topological coarsening of a definable topology is definable was Thm 4.10, so all V-topological coarsenings of the canonical topology τ_{can} are definable. Conversely, if τ is a definable V-topology on K, then there is a definable bounded neighbourhoof $B \in \tau$ such that for every $x, y \in K$, either $x \in By$ or $y \in Bx$ (W_1-top), i.e., for all $x \in K^\times$, either x or $1/x \in B$; so $K = B \cup B^{-1}$, and B must have full rank, whence $B - B \in \tau_{can} \cap \tau^\perp$.

Theorem 6.8. *Assume that whenever K is a dp-finite field of characteristic* 0*, then τ_{can} is generated by independent valuations $\tau_1, \ldots, \tau_n$ (some n), and that each τ_i has a unique V-topological coarsening. Then Shelah's conjecture holds for dp-finite fields.*

It is enough to show that the henselianity conjecture holds, i.e., that each $\tilde{\tau}_i$ is henselian ($\tilde{\tau}_i$ top. coarsening of τ_i). If not, then K has a finite extension L, with some $\tilde{\tau}_i$ having two distinct extensions, whose valuation rings $\mathcal{O}_1$ and $\mathcal{O}_2$ are incomparable. With a little bit of work, one reduces to the situation: K dp-finite, with two independent valuation rings $\mathcal{O}_1$, $\mathcal{O}_2$, and we work in the (dp-finite) structure $(K, \mathcal{O}_1, \mathcal{O}_2)$. We let σ_{can} be the canonical topology on K, with independent generators $\sigma_1, \ldots, \sigma_m$. Then $m > 1$, since otherwise $\sigma_{can} = \sigma_1$ has two distinct V-topological coarsenings. So $m \geq 2$, and one shows that this is impossible. The proof uses char$(K) = 0$, which we may assume.