# dp-finite abelian groups

broad and narrow sets

MSRI dp-finite fields reading group

November 24, 2020
+ December 3, 2020

# Dp-finite 1 - Section 3

1. Introduce the broad filter.

2. Relate broadness to externally definable sets.

3. Relate broadness to dp-rank

# Broad Sets

**Definition**

Let $X_1, \ldots, X_n$ be definable sets and

$Y \subseteq X_1 \times \ldots \times X_n$ be type-definable.

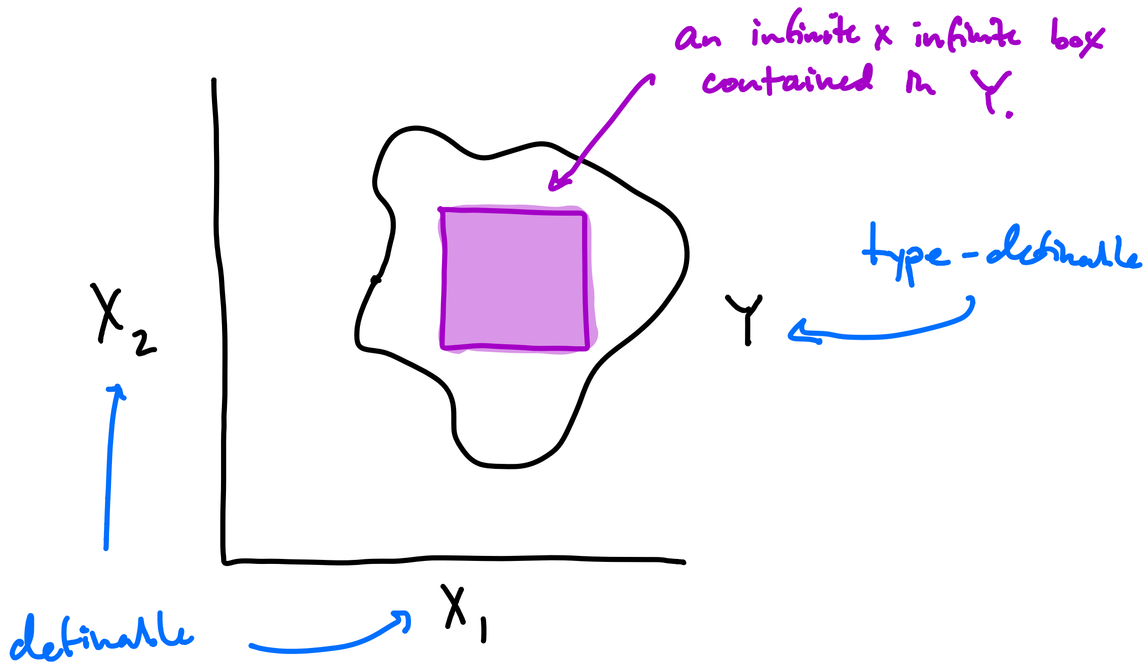
$Y$ is **broad** if there exist

$a_{i,j} \in X_i$ for all $1 \leq i \leq n$, $j \in \mathbb{N}$ such that

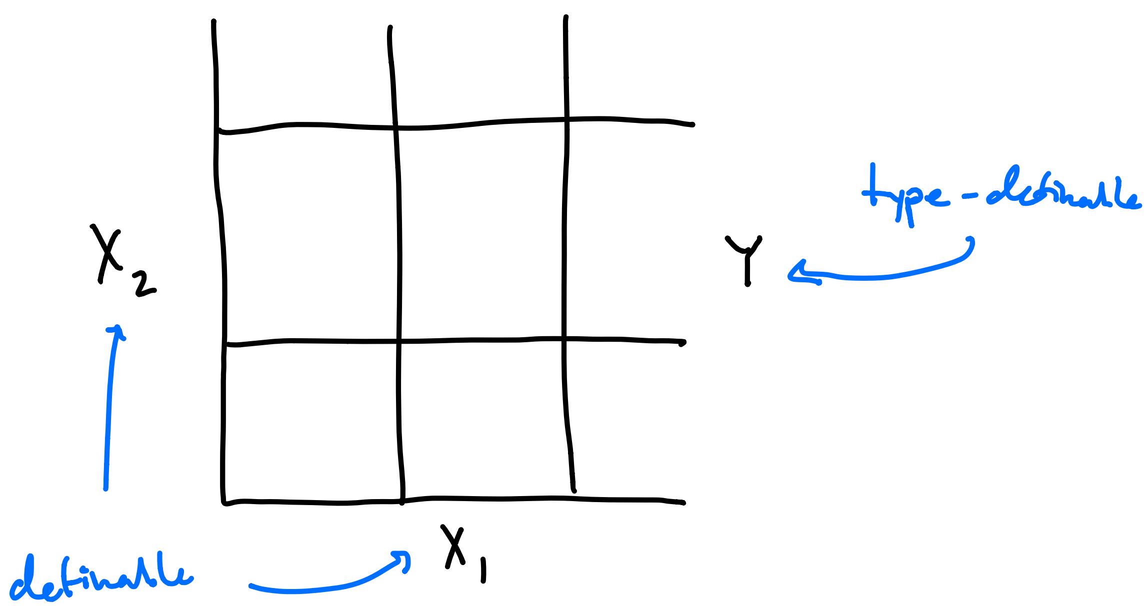
- For all $i$, the $a_{i,j}$ are pairwise distinct (as $j$ varies)
- For any function $\eta : [n] \to \mathbb{N}$

  $(a_{1,\eta(1)}, a_{2,\eta(2)}, \ldots, a_{n,\eta(n)}) \in Y$.

Otherwise, we say $Y$ is **narrow**.




Broad




Narrow

# Easy Observations

(1) $Y \subseteq X_1$ is broad if and only if $Y$ is infinite.

(2) $Y$ is broad if and only if, for all $m \in \mathbb{N}$, there are sets $S_i \subseteq X_i$ with $|S_i| = m$ for all $1 \leq i \leq n$ such that $\prod_{i=1}^{n} S_i \subseteq Y$.

(3) In the definition of broad, we may assume $(a_{i,j})_{\substack{i \in [n] \\ j \in \mathbb{N}}}$ is mutually indiscernible.

$\bar{a}_i$ is indiscernible over $\bar{a}_{\neq i}$ for all $i \in [n]$.

# Type-definability in families

(1) If $\{D_b\}_{b \in Y}$ is a definable family of definable subsets of $X_1 \times \dots \times X_n$, then

$\{b \in Y \mid D_b \subseteq X_1 \times \dots \times X_n \text{ is broad}\}$

is type-definable.

(2) If $A$ is a small set of parameters and $D$ is a definable set. The set of tuples $(a_1, \dots, a_n, b) \in X_1 \times \dots \times X_n \times Y$ such that $tp(\bar{a}/Ab)$ is broad is type-definable.

**Proof**

(1) $D_b$ is broad $\iff$

$$\bigwedge_{m \in \mathbb{N}} \exists (a_{i,j})_{\substack{i \in [n] \\ j \in [m]}} \left( \bigwedge_{\substack{i \in [n] \\ j \neq j' \in [m]}} a_{i,j} \neq a_{i,j'} \wedge \bigwedge_{\eta : [n] \to [m]} (a_{1,\eta(1)}, \dots, a_{n,\eta(n)}) \in D_b \right)$$

(2) $tp(\bar{a}/Ab)$ is broad $\iff$

$$\bigwedge_{m \in \mathbb{N}} \exists (a_{i,j})_{\substack{i \in [n] \\ j \in [m]}} \left( \bigwedge_{\substack{i \in [n] \\ j \neq j' \in [m]}} a_{i,j} \neq a_{i,j'} \wedge \bigwedge_{\eta : [n] \to [m]} (a_{1,\eta(1)}, \dots, a_{n,\eta(n)}) \equiv_{bA} (a_1, \dots, a_n) \right)$$

$\blacksquare$

# The broad filter

Let $X_1, \ldots, X_n$ be infinite definable sets. The broad type-definable subsets of $X_1 \times \cdots \times X_n$ are a filter on type definable subsets of $X_1 \times \cdots \times X_n$, (or, equivalently, the narrow sets are an ideal).

Proof Fix $Y, Z \subseteq X_1 \times \cdots \times X_n$ type-definable. First, note that $Y$ broad and $Y \subseteq Z$ then $Z$ is broad. Secondly, suppose $Y \cup Z$ is broad. As observed above, this entails we can find some mutually indiscernible array $(a_{i,j})_{\substack{i \in [n] \\ j \in \mathbb{N}}}$ over some small set $A$, over which $Y$ and $Z$ are defined,

such that, for all $i \in [n]$, $(a_{i,j})_{j \in \mathbb{N}}$ are pairwise distinct and for all $\eta : [n] \to \omega$,

$$(a_{i,\eta(i)})_{i \in [n]} \in Y \cup Z,$$

hence, by indiscernibility, we have

$$(a_{i,\eta(i)})_{i \in [n]} \in Y$$

for all $\eta : [n] \to \mathbb{N}$ or

$$(a_{i,\eta(i)})_{i \in [n]} \in Z$$

for all $\eta : [n] \to \mathbb{N}$. This shows either $Y$ or $Z$ is broad. $\blacksquare$

# Passage to Complete Types

## Proposition

Let $A$ be a small set of parameters and suppose $X_1, \ldots, X_n$ are $A$-definable sets. Sp. $Y$ type-definable over $A$. Then $Y$ is broad if and only if $tp(a/A)$ is broad for some $a \in Y$.

## Proof

$\Leftarrow$ Clear.

$\Rightarrow$ Extract a mutually indiscernible array $(a_{i,j})_{\substack{i \in [n] \\ j \in \mathbb{N}}}$ over $A$ from a witness to the broadness of $Y$. Consider $tp(a_{1,1}, a_{2,1}, \ldots, a_{n,1}/A)$.

$\blacksquare$

# Slices and Hyperplanes

**Lemma**

Assume NIP. Let $X_1, \ldots, X_n$ be infinite definable sets and $Y \subseteq X_1 \times \cdots \times X_n$ be definable. Assume $Y$ is broad and $n \geq 2$.

1. There is some $b \in X_n$ such that the "slice"

$$\{(a_1, \ldots, a_{n-1}) \in X_1 \times \cdots \times X_{n-1} \mid (a_1, \ldots, a_{n-1}, b) \in Y\}$$

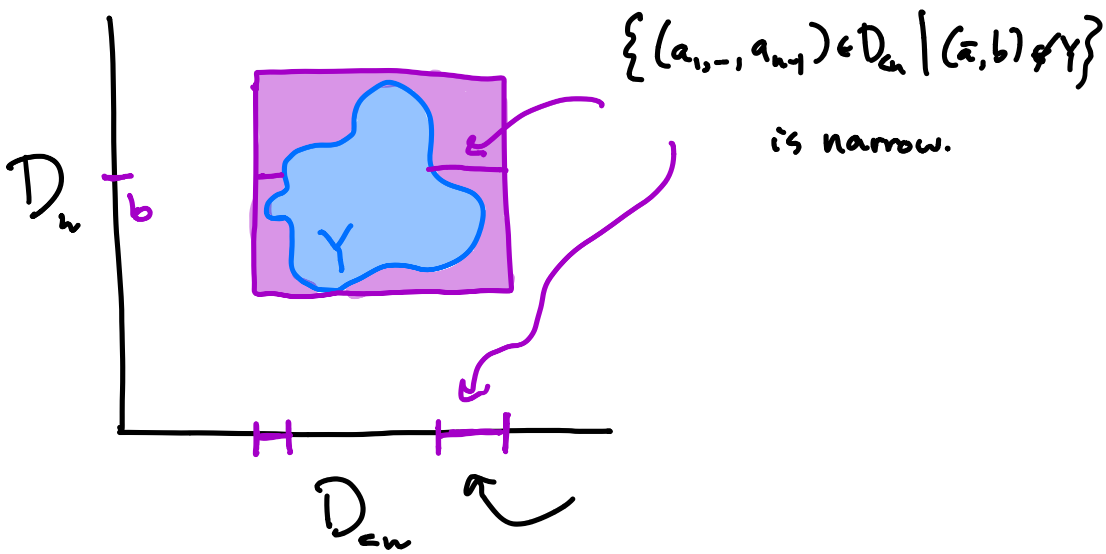
is a broad subset of $X_1 \times \cdots \times X_{n-1}$.

2. There is a broad definable subset $D_{<n} \subseteq X_1 \times \cdots \times X_{n-1}$ and an infinite definable $D_n \subseteq X_n$ such that

$(D_{<n} \times D_n) \setminus Y$ is a hyperplane in the sense that for every $b \in D_n$, the definable set

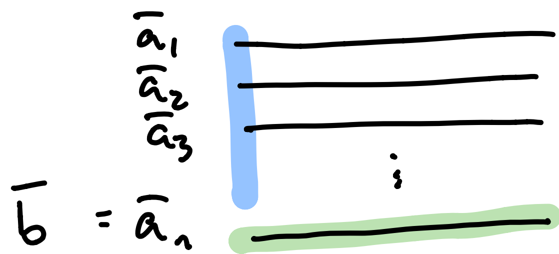
$$\{(a_1, \ldots, a_{n-1}) \in D_{<n} \mid (a_1, \ldots, a_{n-1}, b) \notin Y\}$$

is narrow in $X_1 \times \ldots \times X_{n-1}$.



**Proof** Fix $A$ over which everything is definable. Let $(a_{i,j})_{i\in[n],\, j\in\mathbb{N}}$ be mutually indiscernible over $A$ witnessing $Y$ is broad. Define

$b_j = a_{n,j}$ for all $j<\omega$.




Note $(a_{i,j})_{i\in[n-1],\, j\in\mathbb{N}}$ are mutually indiscernible over $A\bar{b}$, and $\mathrm{tp}(a_{1,1}, a_{2,1}, \dots, a_{n-1,1}/A\bar{b})$ is a broad type in $X_1\times\dots\times X_{n-1}$; also $(a_{1,1},\dots,a_{n-1,1})$ is in the slice

$$\{(a_1,\dots,a_{n-1})\in X_1\times\dots\times X_{n-1} \mid (a_1,\dots,a_{n-1},b_1)\in Y\}$$

which shows (1). [NB Notice this did not use NIP].

For (2), consider sequences $c_1, c_2, \ldots, c_m$ satisfying the following:

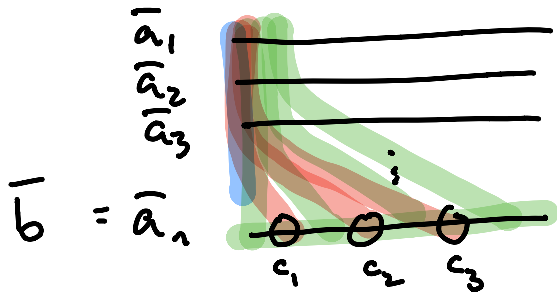
1. $I = b_1 c_1 b_2 c_2 b_3 c_3 \ldots b_m c_m b_{m+1} b_{m+2} b_{m+3} \ldots$

   is $A$-indiscernible

2. $tp(a_{1,1}, \ldots, a_{n-1,1}/AI)$ is broad

3. $(a_{1,1}, \ldots, a_{n-1,1}, c_j) \notin Y$ for $j = 1, \ldots, m$.

The empty sequence is one such sequence, and since $(a_{1,1}, \ldots, a_{n-1,1}, b_j) \in Y$ for all $j$, there is a bound, depending only on $(a_{1,1}, \ldots, a_{n-1,1})$ and $Y$, on the length of a sequence one can build satisfying 1 & 3, by NIP. So let $m$ be maximal.




Fix such a sequence $(c_i)_{i=1}^{m}$ and let $c_{m+1}$ be chosen so that $b_1 c_1 b_2 c_2 \ldots b_m c_m b_{m+1} c_{m+1} b_{m+2} \underset{\sim}{b_{m+3}} \ldots$ is $A$-indiscernible.

**Claim**

If $\bar{\alpha} \in X_1 \times \ldots \times X_{n-1}$ and $\beta \in X_n$ satisfy

- $\bar{\alpha} \equiv_{AI} (a_{1,1}, \ldots, a_{n-1,1})$
- $\beta \equiv_{AI} c_{m+1}$
- $(\bar{\alpha}, \beta) \notin Y$

then $tp(\bar{\alpha}/AI\beta)$ is narrow.

Proof By an automorphism over $AI$, wlog $\bar{\alpha} = (a_{1,1}, \ldots, a_{n-1,1})$. If $tp(\bar{\alpha}/AI\beta)$ is broad, then, by choice of $c_{m+1}$ and the 2nd bullet, we have

$b_1 c_1 b_2 c_2 \ldots b_m c_m b_{m+1} \beta\, b_{m+2} c_{m+2} \ldots$

is $A$-indiscernible, contradicting the maximality of $m$. ∎

Then there are formulas $\varphi(\bar{x}) \in tp(a_{1,1}, \ldots, a_{n-1,1}/AI)$ and $\psi(y) \in tp(c_{m+1}/AI)$ such that

$$\varphi(\bar{\alpha}) \wedge \psi(\beta) \wedge ((\bar{\alpha}, \beta) \notin \Upsilon)$$

entails $tp(\bar{\alpha}/AI\beta)$ is narrow.

Let $D_{<n}$ be the subset of

$X_1 \times \dots \times X_{n-1}$ defined by $\varphi(\bar{x})$, let $D_n$ be the subset of $X_n$ defined by $\psi(y)$. Because $b_1 c_1 b_2 c_2 \dots b_{m+1} c_{m+1} b_{m+2} b_{m+3} \dots$ is indiscernible and non-constant, it follows

$c_{m+1} \notin acl(A\, b_1 c_1 \dots b_m c_m b_{m+1} b_{m+2} \dots)$

Therefore, $D_n$ is infinite. Also $D_{<n}$ is a broad subset of $X_1 \times \dots \times X_{n-1}$ because $tp(a_{1,1} \dots a_{n-1,1}/AI)$ is broad.

Now we have to show that if $\beta \in D_n$ then the set

$$\{ \bar{a} \in D_{<n} \mid (\bar{a}, \beta) \notin Y \}$$

is narrow in $X_1 \times \dots \times X_{n-1}$. But if $\bar{a}$ is in this set, by choice of $\varphi, \psi$, $tp(\bar{a}/A I_{\cancel{\phi}})$ is narrow, so this follows by the 'passage to complete types' lemma. $\blacksquare$

# Hyperplane Theorem

Assume NIP. Let $X_1, \ldots, X_n$ be definable and $Y$ a definable subset of $X_1 \times \ldots \times X_n$. Then $Y$ is broad if and only if there exist infinite subsets $D_i \subseteq X_i$ such that $(D_1 \times \cdots \times D_n) \setminus Y$ is a hyperplane — i.e. for every $b \in D_n$, the set

$$\{(a_1, \ldots, a_{n-1}) \in D_1 \times \cdots \times D_{n-1} \mid (a_1, \ldots, a_{n-1}, b) \notin Y\}$$

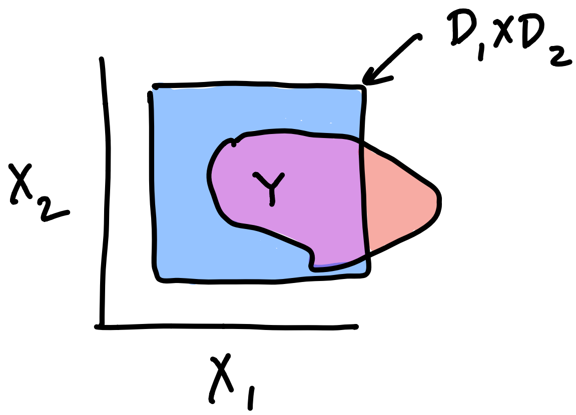
is narrow in $X_1 \times \cdots \times X_{n-1}$.

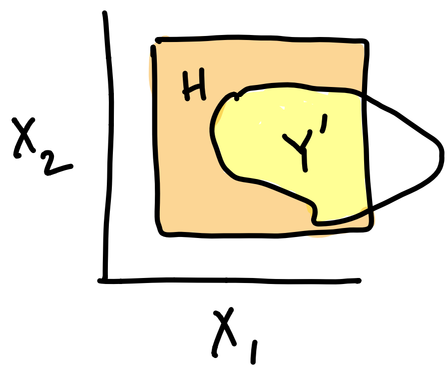
Proof "If" direction:

$D_1 \times \cdots \times D_n$ is clearly broad since each $D_i$ is infinite. Consider the sets

$$Y' := (D_1 \times \dots \times D_n) \cap Y$$

$$H := (D_1 \times \dots \times D_n) \setminus Y$$




As $Y' \cup H = D_1 \times \dots \times D_n$, which is broad, one of $Y'$ and $H$ must be broad.

By the 'slice' lemma, if $H$ is broad, it cannot be a hyperplane. So it follows that $Y'$ is broad. So $Y$ is broad.

Now we prove the "only if" direction. So we assume $Y$ is broad and proceed by induction on $n$. For $n=1$, $D_1 = Y$ works ✓.

So assume $n>1$. By the 'hyperplane lemma', there are sets $D_{<n} \subseteq X_1 \times \dots \times X_{n-1}$, and $D_n \subseteq X_n$ with $D_{<n}$ broad, $D_n$ infinite, and $(D_{<n} \times D_n) \setminus Y$ a hyperplane. By induction, there are infinite def'l sets $D_i \subseteq X_i$ for $i \in [n-1]$ such that

$$H := (D_1 \times \dots \times D_{n-1}) \setminus D_{<n}$$

is a hyperplane. By the slice lemma,

$H'$ is narrow. For any $b \in D_n$,

$$\{\bar{a} \in D_1 \times \dots \times D_n \mid \bar{a} \notin D_{<n}\} = H'$$

$$\{\bar{a} \in D_{<n} \mid (\bar{a}, b) \notin Y\}$$

are both narrow (as $(D_{<n} \times D_n) \setminus Y$ is a hyperplane) so their union is narrow.

But their union contains

$$\{\bar{a} \in D_1 \times \dots \times D_{n-1} \mid (\bar{a}, b) \notin Y\}$$

so this set is narrow too. This show

$D_1 \times \dots \times D_n \setminus Y$ is a hyperplane, as desired.

# Definability in Families

## Theorem (3.11)

Assume NIP and eliminates $\exists^\infty$. Then broadness is definable in families on $X_1 \times \dots \times Y_n$, i.e. if $\{Y_b\}_{b \in Z}$ is a definable family of definable subsets of $X_1 \times \dots \times X_n$, then

$$\{ b \in Z \mid Y_b \text{ is broad} \}$$

is definable.

**Proof** Proof by induction on $n$. For $n=1$, this is equivalent to elimination of $\exists^\infty$.

Assume $n > 1$ and $\{Y_b\}_{b \in Z}$ is a definable family of definable subsets of $X_1 \times \dots \times X_n$.

Let $A$ be a set over which everything is defined. We have proved

$$\{ b \in Z \mid Y_b \text{ is broad} \}$$

is type-definable, so it suffices to show this set is $\bigvee$-definable. Fix $b_0 \in Z$ such that $Y_{b_0}$ is broad.

By the previous theorem, there exist infinite definable sets $D_i = \varphi_i(M; c_i) \subseteq X_i$ such that for all $b \in D_n$

$$\{ (a_1, \ldots, a_{n-1}) \in D_1 \times \ldots \times D_{n-1} \mid (a_1, \ldots, a_{n-1}, b) \notin Y_{b_0} \}$$

is narrow. Reformulating this syntactically, if $b_0 \in Z$ and $Y_{b_0}$ is broad

$$(\forall y \in X_n)\left[\varphi_n(x;c_n) \to \neg\left[\exists\right]^{\text{broad}} (x_1,\dots,x_{n-1}) \in X_1 \times \dots X_{n-1}\right)$$

$$\left(\left((x_1,\dots,x_{n-1},y) \notin Y_{b_0} \wedge \bigwedge_{i=1}^{n-1} \varphi_i(x_i;c_i)\right)\right].$$

Call this set $\psi_{\varphi_1,\dots,\varphi_n}(b_0;c_1,\dots,c_n)$.

Let $\psi'_{\varphi_1,\dots,\varphi_n}(x;z_1,\dots,z_n) = \psi_{\varphi_1,\dots,\varphi_n}(x;z_1,\dots,z_n) \wedge \bigwedge_{i=1}^{n}(\exists^\infty_x)\varphi_i(x;z_i)$.

Note that if $b_0' \in Z$ and for some $c_1',\dots,c_n'$,

$\models \psi'_{\varphi_1,\dots,\varphi_n}(b_0',c_1',\dots,c_n')$ then $Y_{b_0'}$ is also broad.

So $\{b \in Z \mid Y_b \text{ is broad}\}$ is defined by

$$\bigcup_{\varphi_1,\dots,\varphi_n} (\exists z_1,\dots,z_n)\psi'_{\varphi_1,\dots,\varphi_n}(\mathbb{M};z_1,\dots,z_n). \qquad \blacksquare$$

## Corollary of Definability in Families

Assume $T$ is NIP and eliminates $\exists^\infty$. Let $X_1, \dots, X_n$ be definable sets and $\{D_b\}_{b \in Y}$ a definable family of subsets of $X_1 \times \dots \times X_n$. There is some constant $m$, depending on the family such that, for $b \in Y$, the set $D_b$ is broad if and only if there exist $(a_{i,j})_{i \in [n], j \in [m]}$ such that

- For all $i \in [n]$, $a_{i,j} \neq a_{i,j'}$, for $j \neq j'$.
- For any $\eta : [n] \to [m]$, $(a_{1,\eta(1)}, \dots, a_{n,\eta(n)}) \in D_b$.

## Externally definable sets

Assume NIP and eliminates $\exists^\infty$. Let $M \preccurlyeq \mathbb{M}$ be a small model. Let $X_1, \dots, X_n$ be $M$-definable infinite sets and $Y \subseteq X_1 \times \dots \times X_n$ be $M$-definable and broad. Let $D_1, \dots, D_\ell$ be $M$-definable subsets of $X_1 \times \dots \times X_n$ such that

$$Y(M) \subseteq \bigcup_{k=1}^{\ell} D_k.$$

Then there exist some $k$ and some $M$-definable broad set $Y' \subseteq Y$ such that $Y'(M) \subseteq D_k$.

## Proof

As $Y$ is broad, for each $m \in \mathbb{N}$, there is an array $(a_{i,j})_{i \in [n], j \in [m]}$ such that

(a) $a_{i,j} \in X_i$ (or even $X_i(M)$)

(b) $a_{i,j} \neq a_{i,j'}$ for $j \neq j'$

(c) For any $\eta : [n] \to [m]$, $(a_{i,\eta(i)})_{i \in [n]} \in Y$ (or $Y(M)$).

By Ramsey's theorem for convexly partitioned linear orders, there is some $k \in [\ell]$ such that for every $m$, we can find some $(a^{\prime}_{i,j})_{i \in [n], j \in [m]}$ such that

(a) $a_{i,j} \in X_i$

(b) $a_{i,j} \neq a_{i,j'}$ for $j \neq j'$.

(c) For any $\eta : [n] \to [m]$,

$$(a_{i,\eta(i)})_{i \in [n]} \in D_k .$$

# Digression on the Ramsey Statement

Recall that a Fraïssé class $\mathbb{K}$ of finite (rigid) L-structures is called a Ramsey class if for all $A \leq B \in \mathbb{K}$ and $r < \omega$, is $C \in \mathbb{K}$ such that

$$C \longrightarrow (B)^A_r, \quad \text{i.e.}$$

for all $f : \binom{C}{A} \to [r]$, there is $B' \in \binom{C}{B}$ such that $f \restriction \binom{B'}{A}$ is constant, where

$$\binom{X}{Y} = \{ Y' \subseteq X \mid Y' \cong Y \}.$$

Let $L_{P_1,\dots,P_n} = \{\leq, P_1, \dots, P_n\}$ where each $P_i$ is a unary predicate. Let $\mathbb{K}_{P_1,\dots,P_n}$ be the class of finite $L$-structures in which $\leq$ is interpreted as a linear order, the $P_i$'s are a partition, and $P_i < P_j$ for $i < j$. This is a Fraïssé class.

**Exercise**

$\mathbb{K}_{P_1,\dots,P_n}$ is a Ramsey class.

**Hint:** Ramsey's theorem.

Then let $A = \{1, \dots, n\}$ with $P_i^A = \{i\}$ and $\leq^A$ interpreted naturally.

Let $B_m = \{(i,j) : i \in [n], j \in [m]\}$ with $\leq^{B_m}$ interpreted lexicographically and

$$P_i^{B_m} = \{(i,j) : j \in [m]\}$$

As every structure in $\mathbb{K}_{P_1, \dots, P_n}$ is isomorphic to a substructure of one of the $B_m$'s, the Ramsey property for $\mathbb{K}_{P_1, \dots, P_n}$ implies: for every $m \in \mathbb{N}$, there is $N(m) \in \mathbb{N}$ such that

$$B_{N(m)} \rightarrow (B_m)^A_k.$$

Note that if we are given some $A' \in \binom{B_m}{A}$, there is

some $\eta_{A'} : [n] \to [m]$ such that

$$A' = \{ \eta_{A'}(1), \ldots, \eta_{A'}(n) \} \text{ with}$$

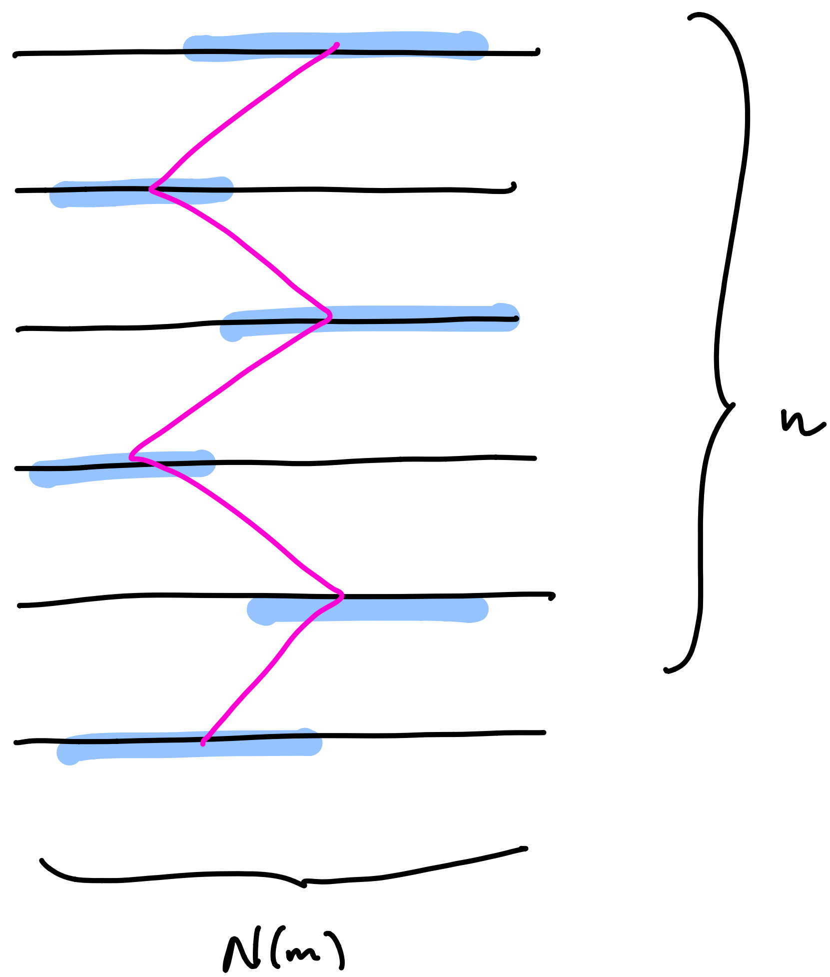
$$P_i^{A'} = \{ \eta_{A'}(i) \}.$$

Given $m \in \mathbb{N}$, define a coloring

$$\chi_m : \binom{B_{N(m)}}{A} \longrightarrow [k]$$

by

$$\chi_m(A') = \min \left\{ j \,\middle|\, (a_{i,\eta_{A'}(i)})_{i \in [n]} \in D_j \right\}.$$



n
N(m)

By the Ramsey property, there is some $B' \in \binom{B_{N(m)}}{B_m}$ such that $\chi_m \restriction \binom{B'}{A}$ is constant with value $j(m)$.

Then defining $(a'_{i,j})_{i \in [n], j \in [m]}$ so that $(a'_{i,j})_{j \in [m]}$ is an increasing enumeration of $P_i^{B'}$, we have that for any $\eta : [n] \to [m]$,

$$(a'_{i,\eta(i)})_{i \in [n]} \in D_{j(m)}.$$

By the pigeonhole principle,

there is some $k$ such that
for infinitely many $m \in \mathbb{N}$,

$$j(m) = k.$$

This is the $k$ we are looking for. ∎

— end of

digression —

By honest definitions, the externally definable set $Y(M) \cap D_k$ can be approximated by internally definable sets as follows: there is an $M$-definable family $\{F_b\}$ such that, for every finite subset $S \subseteq Y(M) \cap D_k$, there is $b \in M$ such that

$$S \subseteq F_b(M) \subseteq Y(M) \cap D_k.$$

Take $m$ as in the Corollary of definability of families, for the family $\{F_b\}$.

Take $(a_{i,j})_{\substack{i \in [n] \\ j \in [m]}}$ such that

- $a_{i,j} \in X_i(M)$
- $a_{i,j} \neq a_{i,j'}$ for $j \neq j'$.
- For any $\eta : [n] \to [m]$, the tuple $(a_{1,\eta(1)}, \ldots, a_{n,\eta(n)}) \in D_k \cap Y(M)$.

Let $S = \{(a_{1,\eta(1)}, \ldots, a_{n,\eta(n)}) : \eta : [n] \to [m]\}$.

Pick $b$ such that

$$S \subseteq F_b(M) \subseteq Y(M) \cap D_k.$$

Then $F_b$ is broad. Let $Y' = F_b$.

Then $Y'(M) \subseteq Y(M)$ implies $Y' \subseteq Y$

and $Y' \subseteq D_k$. ∎

## The Finite Rank Setting

### Lemma 3.15

Let $X_1, \ldots, X_n$ be $A$-definable sets, and let $(a_1, \ldots, a_n) \in X_1 \times \cdots \times X_n$. Suppose there is an infinite sequence $(b_i)_{i \in \mathbb{N}}$ of pairwise distinct elements such that

$$(a_1, \ldots, a_{n-1}, b_i) \equiv_A (a_1, \ldots, a_n)$$

for all $i \in \mathbb{N}$, and $tp(a_1, \ldots, a_{n-1} / A\bar{b})$ is broad, then $tp(a_1, \ldots, a_n / A)$ is broad.

**Proof** Let $(e_{i,j})_{\substack{i\in[n-1]\\ j\in\mathbb{N}}}$ witness that $tp(a_1,\dots,a_{n-1}/A\bar{b})$ is broad. Then for all $\eta:[n-1]\to\mathbb{N}$, we have

$$(e_{1,\eta(1)},\dots,e_{n-1,\eta(n-1)})\equiv_{A\bar{b}}(a_1,\dots,a_{n-1})$$

hence

$$(e_{1,\eta(1)},\dots,e_{n-1,\eta(n-1)},b_i)\equiv_A(a_1,\dots,a_{n-1},b_i)$$

$$\equiv_A(a_1,\dots,a_{n-1},a_n)$$

for all $i$. Then, setting $e_{n,j}=b_j$ for all $j\in\mathbb{N}$ yields an array $(e_{i,j})_{\substack{i\in[n]\\ j\in\mathbb{N}}}$ witnessing that $tp(a_1,\dots,a_n/A)$ is broad. ∎

# Will's Remark

Suppose $X_1, \ldots, X_n$ are $A$-definable sets with $\text{dp-rk}(X_i) = r_i$. Then if $(a_1, \ldots, a_n) \in X_1 \times \cdots \times X_n$, then

$$\text{dp-rk}(a_1, \ldots, a_n / A) \leq r_1 + \cdots + r_n,$$

by sub-additivity of dp-rank and, if

$$\text{dp-rk}(a_1, \ldots, a_n / A) = r_1 + \cdots + r_n,$$

then $\text{dp-rk}(a_i / A a_1 \ldots a_{i-1} a_{i+1} \ldots a_n) = r_i$.

Otherwise, for some $i$, we have

$$\text{dp-rk}(a_i / A a_1 \ldots a_{i-1} a_{i+1} \ldots a_n) < r_i,$$

$$\Rightarrow r_1 + \cdots + r_n \leq \text{dp-rk}(a_i / A a_1 \ldots a_{i-1} a_{i+1} \ldots a_n) + \text{dp}(a_1 \ldots a_{i-1} a_{i+1} \ldots a_n / A)$$

$$< r_i + (r_1 + \cdots + r_{i-1} + r_{i+1} + \cdots + r_n), \text{ contradiction}$$

$\blacksquare$

## First connection between rank and breadth

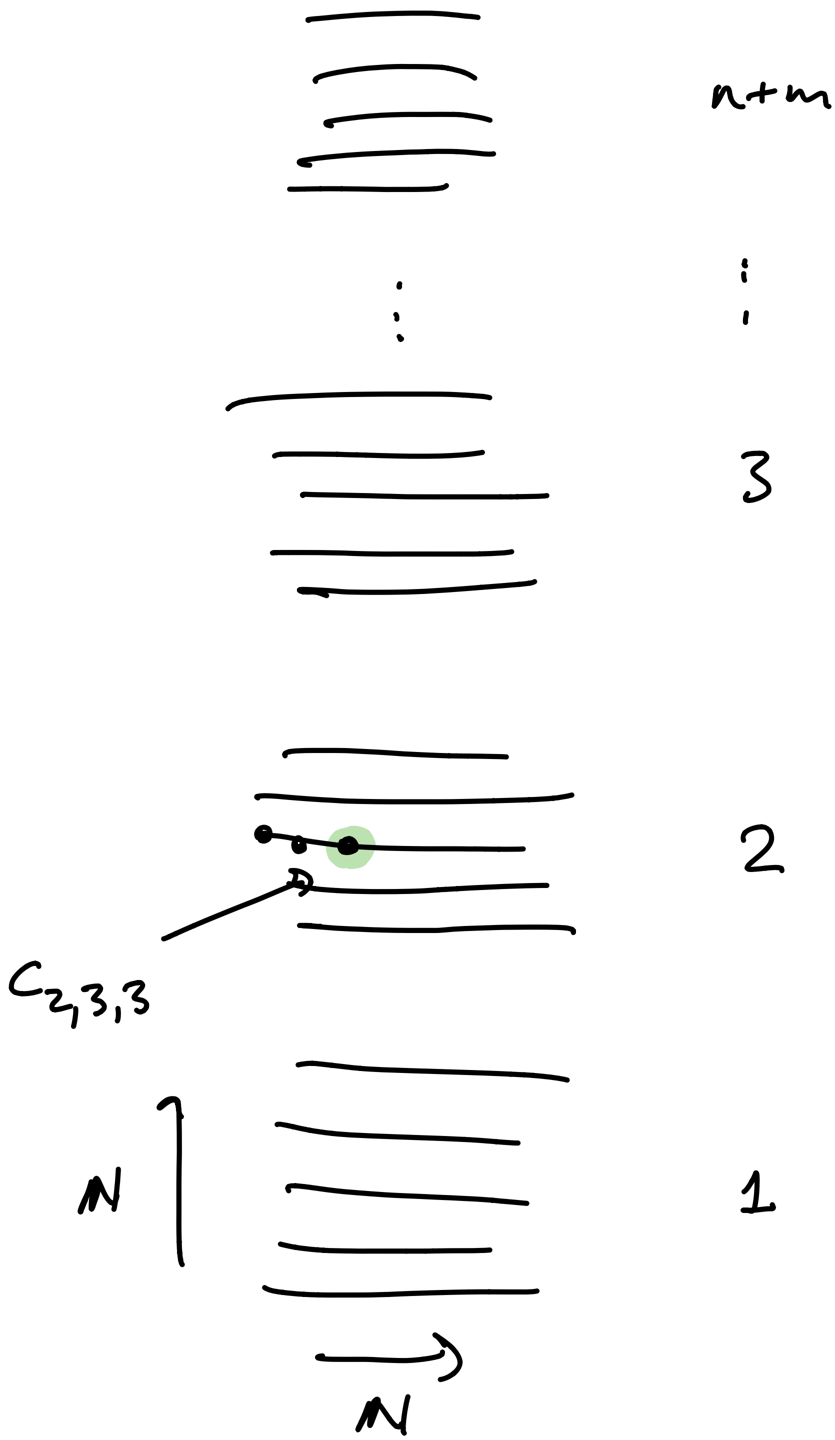
**Lemma 3.17**

Let $X, Y$ be infinite $A$-definable sets of finite dp-rank $n$ and $m$ respectively. Let $(a,b) \in X \times Y$ be a tuple with $\text{dp-rk}(a,b/A) = n+m$. Then there is a sequence of pairwise distinct elements $(b_i)_{i\in\mathbb{N}}$ such that $ab_i \equiv_A ab$ and such that $\text{dp-rk}(a/Ab_1 \ldots b_\ell) = n$ for all $\ell \in \mathbb{N}$.

**Proof** Let $(\varphi_i(x,y;z_i))_{i\in[n+m]}$ and $(c_{i,j})_{\substack{i\in[n+m]\\ j\in\mathbb{N}}}$ form an ict-pattern of depth $n+m$ in $tp(ab/A)$. By compactness and Ramsey, we can assume that in fact our parameters are

$$(c_{i,j,k})_{\substack{i\in[n+m]\\ j\in\mathbb{N}\times\mathbb{N}}}$$

which forms a mutually indiscernible array over $A$, where $\mathbb{N}\times\mathbb{N}$ is ordered lexicographically.



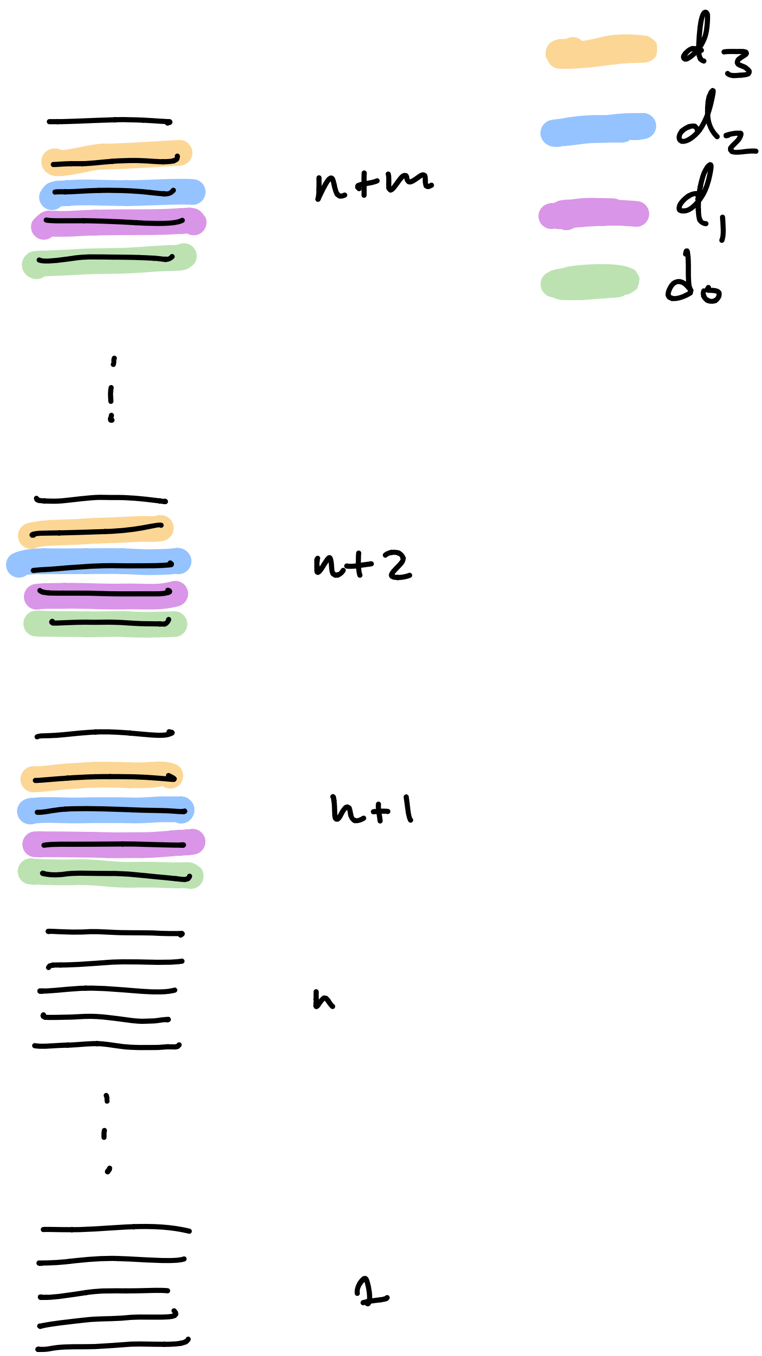
n+m
3
2
$C_{2,3,3}$
N
1
N

So we have

$$\mathrm{tp}(ab/A) \cup \{\varphi_i(x,y;c_{i,0,0}) : i \in [n+m]\}$$
$$\cup \{\neg\varphi_i(x,y;c_{i,j,k}) : (j,k) \neq (0,0)\}.$$

is consistent.

By possibly moving $(c_{i,j,k})_{\substack{i\in[n+m]\\(j,k)\in\mathbb{N}\times\mathbb{N}}}$ over $A$, we may assume this type is realized by $ab$. By the (proof of) sub-additivity of dp-rk, there are $m$ rows which form a mutually indiscernible array over $Aa$. Wlog, these are $i = n+1, \dots, n+m$. For each $j \in \mathbb{N}$, let $d_j$ be a tuple enumerating $(e_{i,j,k})_{\substack{n<i\leq n+m\\k\in\mathbb{N}}}$.


$d_3$
$d_2$
$d_1$
$d_0$
$n+m$
$n+2$
$n+1$
$n$
$1$

The sequence $(d_j)_{j\in\mathbb{N}}$ is an $Aa$-indiscernible sequence.

For each $j \in \mathbb{N}$, pick some $b_j$ with

$$b_j d_j \equiv_{Aa} b d_0.$$

Claim

For any $l > 0$,

$$\text{dp-rk}\left(a b_0 \ldots b_{l-1}/A\right) = n + lm.$$

Proof of Claim

"$\leq$" by sub-additivity of dp-rk.

So suffices to exhibit $(n+lm)$ sequences forming a mutually indiscernible array over $A$, but with none of which is indiscernible over $A a b_0, \ldots, b_{\ell-1}$.

- For $1 \leq i \leq n$, the sequence

  $$(c_{i,0,k})_{k \in \mathbb{N}}$$

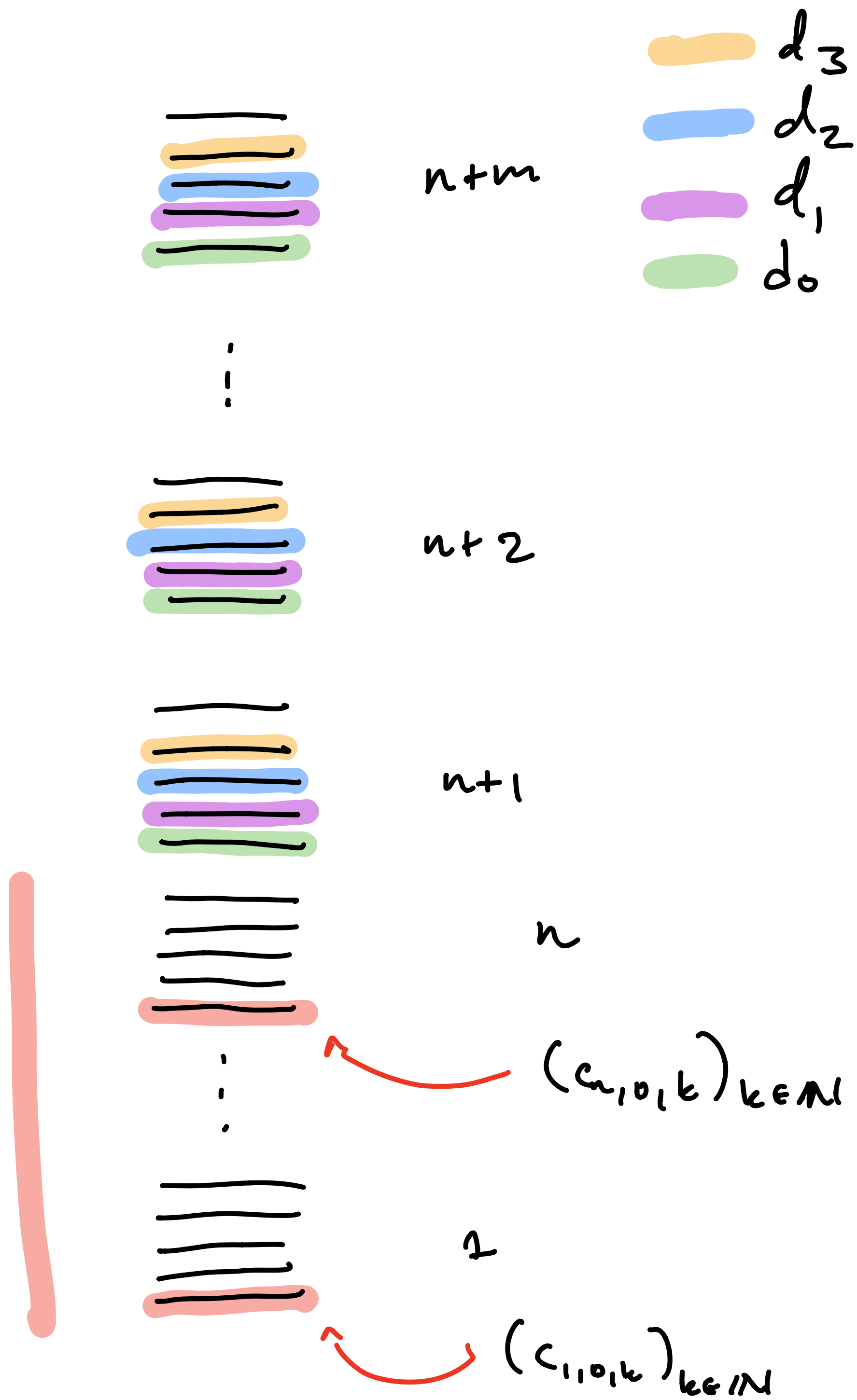
  fails to be indiscernible over

  $ab = ab_0$ since

  $$\models \varphi_i(a, b_0; c_{i,j,k}) \Leftrightarrow (j,k) = (0,0).$$

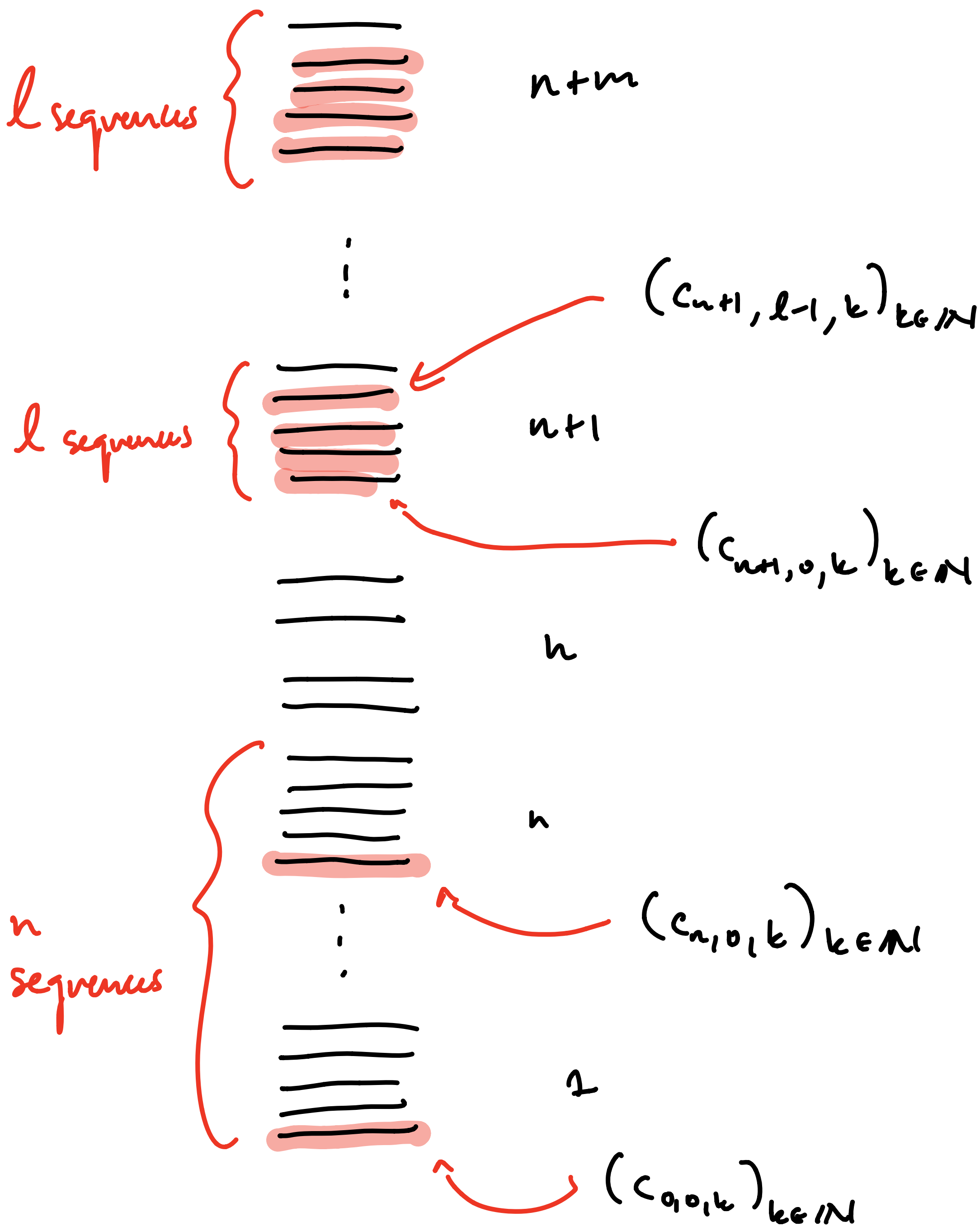
hence

$$\models \varphi_i(a; b_0; c_{i,0,k}) \Leftrightarrow k = 0.$$


$d_3$
$d_2$
$d_1$
$d_0$
$n+m$
$n+2$
$n+1$
$n$
$(c_{n,0,k})_{k\in\mathbb{N}}$
$1$
$(c_{1,0,k})_{k\in\mathbb{N}}$

- For $n < i \leq n+m$, and for $0 \leq j < \ell$, the sequence $(c_{i,j,k})_{k \in \mathbb{N}}$ fails to be indiscernible over $ab_j$ because

$$\models \varphi_i(a, b_j; c_{i,j,k}) \iff \models \varphi_i(a, b; c_{i,0,k})$$
$$\iff k = 0.$$




$n+m\ell$ sequences in total.

These sequences form a mutually indiscernible array by the mutual indiscernibility of $(c_{i,j,k})_{i \in [n+m], (j,k) \in \mathbb{N} \times \mathbb{N}}$. $\blacksquare_{\text{Claim}}$

Now to conclude, we apply the remark to deduce

$$\text{dp-rk}(a/Ab_0b_1 \dots b_{\ell-1}) = n,$$

for each $\ell > 1$. Moreover, by the same, $\text{dp-rk}(b_{\ell-1}/Aab_0 \dots b_{\ell-2}) = m > 0$ so $b_\ell \notin \text{acl}(Aab_0 \dots b_{\ell-1})$, so the $b_j$ are pairwise distinct. $\blacksquare$

# Dp-rank $\Rightarrow$ broad implication

**Proposition 3.19**

For each $i=1,\dots,n$ let $X_i$ be a definable of finite dp-rank $r_i > 0$. Let $Y \subseteq X_1 \times \dots \times X_n$ be type-definable.

If $\text{dp-rank}(Y) = r_1 + \dots + r_n$, then $Y$ is broad.

**Proof** Proof by induction on $n$.

**$n=1$:** $dp(Y) = r_1 > 0 \Rightarrow Y$ infinite
$\Rightarrow Y$ is broad.

Suppose $n > 1$. Let $A$ be a small set of parameters such that everything is over $A$.

Pick $(a_1, \ldots, a_n) \in Y$ such that

$$\text{dp-rk}(a_1, \ldots, a_n / A) = r_1 + \ldots + r_n .$$

By the technical lemma, there is a sequence $(b_j)_{j \in \mathbb{N}}$ of pairwise distinct elements such that

$$(a_1, \ldots, a_{n-1}, b_j) \equiv_A (a_1, \ldots, a_{n-1}, a_n)$$

and

$$\text{dp-rk}(a_1, \ldots, a_{n-1} / A b_1, \ldots, b_j) = r_1 + \ldots + r_{n-1}$$

for all $j \in \mathbb{N}$. By induction, $\text{tp}(a_1 \ldots a_{n-1} / A b_1 \ldots b_j)$ is broad for all $j$, hence $\text{tp}(a_1, \ldots, a_{n-1} / A \bar{b})$ is broad. Hence $\text{tp}(a_1, \ldots, a_n / A)$ is broad, by our first lemma.

This shows $Y$ is broad. $\blacksquare$

# Quasi-minimality

**Definition**

A definable set $D$ is called quasi-minimal if $D$ has finite dp-rank $n > 0$, and every definable subset $D' \subseteq D$ has either rank $0$ or $n$. Equivalently, every infinite definable subset of $D$ has rank $n$.

**Observation 1**: dp-rk $1 \Rightarrow$ quasiminimal

**Observation 2**: Every infinite definable set of finite dp-rk has an infinite quasi-minimal definable subset.

# Main Theorem

Assume NIP. Let $X_1, \dots, X_n$ be quasi-minimal definable sets of rank $r_1, \dots, r_n$ respectively. Let $Y \subseteq X_1 \times \dots \times X_n$ be definable. Then $Y$ is broad if and only if $\text{dp-rk}(Y) = r_1 + \dots + r_n$.

## Proof

By definition, $r_i > 0$ for all $i \in [n]$.

"if" direction was done above.

"only if" will be proven by induction on $n$.

### $n=1$

$$Y \subseteq X_1 \text{ is broad} \Leftrightarrow Y \text{ is infinite} \overset{\text{quasi-minimality}}{\Leftrightarrow} \text{dp-rk}(Y) = r_1.$$

Now assume $n > 1$, and let $Y$ be broad.

By the hyperplane lemma, there are infinite definable sets $D_1 \subseteq X_1, \dots, D_n \subseteq X_n$ such that for every $b \in D_n$, the set

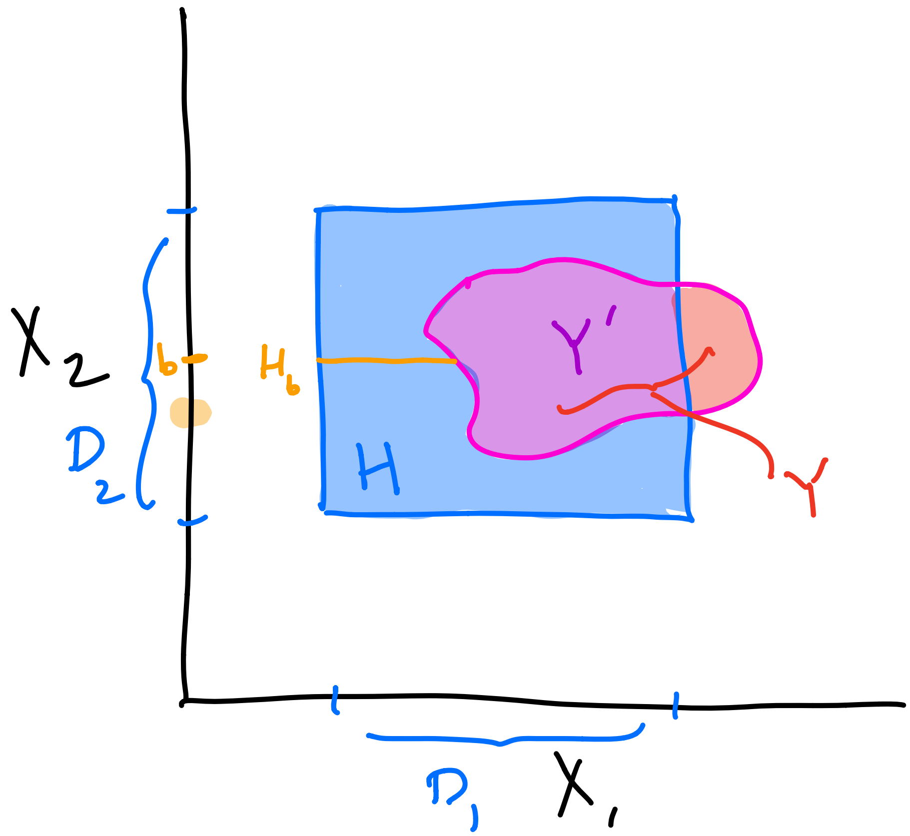
$$H_b := \{(a_1, \dots, a_{n-1}) \in D_1 \times \dots \times D_{n-1} \mid (a_1, \dots, a_{n-1}, b) \notin Y\}$$

is narrow, as a subset of $X_1 \times \dots \times X_{n-1}$.

By quasi-minimality, $\text{dp-rk}(D_i) = r_i$, for all $i \in [n]$. Define $H$ and $Y'$ by

$$H = (D_1 \times \dots \times D_n) \setminus Y = \coprod_{b \in D_n} H_b.$$

$$Y' = (D_1 \times \dots \times D_n) \cap Y.$$




By induction,

$$\text{dp-rk}(H_b) < r_1 + \cdots + r_{n-1}$$

for all $b \in D_n$. By sub-additivity,

$$\text{dp-rk}(H) < r_1 + \cdots + r_{n-1} + \text{dp-rk}(D_n)$$

$$= r_1 + \cdots + r_n.$$

But $D_1 \times \dots \times D_n = H \cup Y'$ and

$$\text{dp-rk}(D_1 \times \dots \times D_n) = r_1 + \dots + r_n$$

so $\text{dp-rk}(Y') = r_1 + \dots + r_n$

so $\text{dp-rk}(Y) = r_1 + \dots + r_n.$ $\blacksquare$

# Definability in Families

Assume $T$ is NIP and eliminates $\exists^\infty$.

Let $X_1, \ldots, X_n$ be quasi-minimal definable sets of finite dp-rank.

Let $r = \text{dp-rk}(X_1 \times \cdots \times X_n)$.

Given a definable family $\{D_b\}_{b \in Y}$ of subsets of $X_1 \times \cdots \times X_n$, the set of $b$ such that $\text{dp-rk}(D_b) = r$ is definable, and there is some $m$, even, such that $\text{dp-rk}(D_b) = r$ if and only if there are $S_i \subseteq X_i$ with $|S_i| = m$ for $i \in [n]$ such that $S_1 \times \cdots \times S_n \subseteq D_b$.