A basis of neighbourhoods of 0 in fields of finite dp-rank (from: Will Johnson DpI, section 4)

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Let $\mathfrak{M}$ be a sufficiently saturated dp-finite field and $\mathcal{M}$ be a small elementary substructure of $\mathfrak{M}$.

Aim: Find a definable basis of neighbourhoods of 0 in $\mathfrak{M}$ inducing on $\mathfrak{M}$ a field topology. Then use it to define M-infinitesimals.

Candidates of basic neighbourhood of 0: (Definition 6.3)
$X -_\infty X := \{\delta \in \mathfrak{M} \colon X \cap (X + \delta) \text{ is heavy}\}$, where X is a definable *heavy* subset of $\mathfrak{M}$ (Definition 4.19).

(In particular such subsets of $\mathfrak{M}$ will again be heavy. Properties of these neighbourhoods will be stated in Proposition 6.5. In particular when $\mathfrak{M}$ is not of finite Morley rank we will get a Hausdorff topology.)

(In the dp-minimal case, one chose X to be a infinite, definable subset of $\mathfrak{M}$ and
$X -_\infty X := \{y \in \mathfrak{M} \colon \exists^\infty x_1 \in X \exists x_2 \in X \ \ y = x_1 - x_2\}$.)

dp-rank

As in the previous section, we will use the following properties of the dp-rank (on definable subsets):

1. the dp-rank is subadditive: $dp-rank(ab/A) \leq dp-rank(a/Ab) + dp-rank(b/A)$,
2. the dp-rank of a (finite) cartesian product is the sum of the dp-rank of the factors,
3. the dp-rank of a definable set is 0 iff this set is finite,
4. if f is a definable function and X a definable set, then $dp-rank(X) \geq dp-rank(f(X))$, and $dp-rank(X) = dp-rank(f(X))$ if the fibers are finite,
5. dp-rank$(X \cup Y) = \max\{$dp-rank(X),dp-rank(Y)$\}$.

From now on we will denote dp-rank, by simply rk.

Recall also that since $\mathfrak{M}$ is NIP, $Th(\mathfrak{M})$ eliminates $\exists^\infty$.

The first goal will be to define heavy sets.
In order to do that we need to define critical sets and to define critical sets,
we need the notion of coordinate configuration and their targets.

We follow the numbering of DpI.

Note that in a number of results we will only use the additive group structure on $\mathfrak{M}$.

Coordinate configuration

Let X be a definable subset of dp-rank $n > 0$. Then X is quasi-minimal if any infinite definable subset of X has dp-rank n.
(Recall that quasi-minimal sets exist (Remarks 3.21 and 3.22).)

A coordinate configuration is a tuple $(X_1, \ldots, X_n, P)$, with
$X_i \subset \mathfrak{M}$, $1 \leq i \leq n$, quasi-minimal,
P a broad definable subset of $X_1 \times \ldots \times X_n$ and
the map $\pi : P \mapsto \mathfrak{M} : (x_1, \ldots, x_n) \mapsto x_1 + \ldots + x_n$ has finite fibers.

Recall that P broad means that there are $(a_{ij}) \in X_i$, $1 \leq i \leq n$, $j \in \mathbb{N}$, such that a_{ij}, $j \in \mathbb{N}$, are pairwise distinct for fixed i and for any function $\eta : \{1, \ldots, n\} \to \mathbb{N}$, $(a_{1\eta(1)}, \ldots, a_{n\eta(n)}) \in P$ (Definition 3.1).
(This was defined for type-definable sets).

Targets

Let $(X_1, \ldots, X_n, P)$ be a coordinate configuration.
By Theorem 3.23, P is broad iff $rk(P) = \sum_{i=1}^{n} rk(X_i)$.
Since π has finite fibers, we have $rk(P) = rk(\pi(P))(\leq rk(\mathfrak{M}))$.

The target of $(X_1, \ldots, X_n, P)$ is the image of π where
$\pi : P \to \mathfrak{M} : (x_1, \ldots, x_n) \mapsto x_1 + \ldots + x_n$ and its rank is

$$\sum_{i=1}^{n} rk(X_i).$$

Example: Letting X be a quasi-minimal subset of $\mathfrak{M}$, then (X, X) is a coordinate configuration with target X.

Critical rank and critical sets

Let $(X_1, \ldots, X_n, P)$ be a coordinate configuration and let $\pi : P \to \mathfrak{M} : (x_1, \ldots, x_n) \mapsto x_1 + \ldots + x_n$

The critical rank is the maximum rank of any coordinate configuration (in any power of $\mathfrak{M}$).
(Note that this rank is always smaller than the rank of $\mathfrak{M}$.)

A set Y is critical if it is the target of any coordinate configuration of critical rank.
Note that there are critical subsets of $\mathfrak{M}$ and since $\mathfrak{M}$ is infinite, the critical rank is always > 0.

Proposition (Proposition 4.3(1))

Let $(X_1, \ldots, X_n, P)$ be a configuration with target Y and $\pi : P \to \mathfrak{M} : (x_1, \ldots, x_n) \mapsto x_1 + \ldots + x_n$.
Let $\{D_b\}$ be a definable family of definable subsets of Y, then $\{b \colon rk(D_b) = rk(Y)\}$ is definable.

Proof.

Recall that $r = \sum_i rk(X_i) = rk(Y)$
The fibers of π are finite by assumption and for $D_b \subset Y$, we have that $rk(\pi^{-1}(D_b)) = rk(D_b)$.
By Corollary 3.24 (there, the subsets were in $X_1 \times \ldots \times X_n$), $\{b \colon rk(\pi^{-1}(D_b)) = r\}$ is definable. □

Some properties of critical sets

Let ρ be the critical rank. Let Y be the target of $(X_1, \ldots, X_n, P)$.

1. If Y is a critical set (of rank ρ) and if $Y' \subset Y$ with $rk(Y') = \rho$, then Y' is critical.
   We note that $(X_1, \ldots, X_n, \pi^{-1}(Y'))$ is a coordinate configuration for Y'. Indeed since the fibers of π are finite, $rk(Y') = rk(\pi^{-1}(Y')) = \rho$. So $\pi^{-1}(Y')$ is broad by Theorem 3.23.

2. If Y is critical and $\alpha \in \mathfrak{M}$, then $\alpha + Y$ is also critical.
   We consider the configuration $(X_1 + \alpha, X_2, \ldots, X_n, P')$, where $P' := \{(a_1 + \alpha, a_2, \ldots, a_n) : (a_1, \ldots, a_n) \in P\}$. The map $x \mapsto \alpha + x$ is a definable bijection, so $rk(X_1) = rk(X_1 + \alpha)$ and $X_1 + \alpha$ is still quasi-minimal. Since P is broad, $rk(P) = \sum_{i=1}^{n} rk(X_i) = rk(X_1 + \alpha) + \sum_{i=2}^{n} rk(X_i)$. We have $rk(Y) = rk(Y + \alpha) = rk(P')$, so P' is still broad.
   So $\alpha + Y$ is the target of the configuration $(X_1 + \alpha, X_2, \ldots, X_n, P')$, of critical rank.

Heavy sets

Let Y be a critical set. Then X is Y-heavy if there is $\delta \in \mathfrak{M}$ such that rk$(Y \cap (X + \delta))$ =rk(Y).
In other words, some translate of X has a "big" intersection with Y.

A set X is heavy if it is Y-heavy for every/any critical set $Y \subset M$ (see Proposition below).
In particular, X is heavy iff some translate of X contains a critical set.
M is heavy since it contains a critical set.

A set is light if it is not heavy.

Proposition (Proposition 4.18)

Let Y, Y' be two critical sets and let X be a definable subset of M. Then X is Y-heavy iff X is Y'-heavy.

The proof of that proposition will use a series of auxiliary results.
Denote $[n] := \{1, \ldots, n\}$.

Lemma (Lemma 4.4)

Let $X_1, \ldots, X_n$ be A-definable and quasi-minimal. Let $\bar{a} := (a_1, \ldots, a_n) \in X_1 \times \ldots \times X_n$ such that $tp(\bar{a}/A)$ is broad. Set $s = a_1 + \ldots + a_n$. If $\bar{a} \in acl(sA)$, then there is a broad set $P \subset X_1 \times \ldots \times X_n$ with $\bar{a} \in P$ and $(X_1, \ldots, X_n, P)$ is a coordinate configuration with $s \in \pi(P)$.

Proof.

Recall that $tp(\bar{a}/A)$ is broad if the following holds: $\bigwedge_{m \in \mathbb{N}}$

$$\exists a_{1,1} \ldots a_{n,m}(\bigwedge_{i \in [n], 1 \leq j \leq j' \leq m} a_{i,j} \neq a_{i,j'} \ \wedge$$

$$\bigwedge_{\eta : [n] \to [m]} (a_{1,\eta(1)}, \ldots, a_{n\eta(n))\equiv_A}(a_1, \ldots, a_n)).$$

□

Since $\bar{a} \in acl(sA)$, there is an A-formula $\varphi(\bar{x}, y)$ such that $\varphi(\bar{a}, s)$ holds and $\varphi(\mathfrak{M}, s)$ has cardinality $k \in \mathbb{N}$. Furthermore we may assume that for all $y \quad |\varphi(\mathfrak{M}, y)| \leq k$ and that it contains the subformula $y = \sum_{i=1}^{n} x_i$. Define

$$P := \{\bar{x} \in X_1 \times \ldots \times X_n \colon \varphi(\bar{x}, \sum_{i=1}^{n} x_i)\}.$$

Since P contains $\bar{a}$, it is broad
and the map $\pi : P \to \mathfrak{M} : \bar{x} \mapsto \sum_{i=1}^{n} x_i$ has fibers of size $\leq k$.
So $(X_1, \ldots, X_n, P)$ is a coordinate configuration.

Lemma (Lemma 4.5)

Let $X_1, \ldots, X_n$ be A-definable and quasi-minimal. Let p_i be a global A-invariant type in X_i, $1 \leq i \leq n$. Assume that p_i is not realized. Then the type $p_1 \otimes \ldots \otimes p_n \restriction A$ is broad.

Proof.

Since p_i is A-invariant, we may define $p_i^{\otimes\omega}$ ($\otimes$ is associative). Note that $p_i^{\otimes\omega}$again A-invariant) and a realization is an A-indiscernible sequence. Since p_i is non algebraic the elements of that sequence are pairwise distinct. Then take $(a_{i,j})_{i\in[n],j\in\mathbb{N}}$ be a realization of $p_1^{\otimes\omega} \otimes \ldots \otimes p_n^{\otimes\omega}$ over A.

Then the tuple $(a_{1,\eta(1)}, \ldots, a_{n,\eta(n)})$ is a realization of $p_1 \otimes \ldots \otimes p_n$ over A and the rows of the array $(a_{i,j})_{i\in[n],j\in\mathbb{N}}$ is composed of distinct elements. So, $p_1 \otimes \ldots \otimes p_n$ is broad over A. □

For reference for products of A-invariant types, see [P. Simon, A guide to NIP theories, 2.2.1].

Lemma (Lemma 4.6)

Let $(X_1, \ldots, X_n, P)$ be a coordinate configuration. Then there exist a small model M and non-algebraic global M-invariant types p_i on X_i such that if $\bar{a} \models p_1 \otimes \ldots \otimes p_n \upharpoonright M$, then $\bar{a} \in P$.

Proof.

First we use Theorem 3.10 to find $D_i \subset X_i$, $1 \leq i \leq n$, infinite definable sets such that for every $b \in D_n$

$$\{(u_1, \ldots, u_{n-1}) \in D_1 \times \ldots \times D_{n-1} \colon (u_1, \ldots, u_{n-1}, b) \notin P\}$$

is not a broad subset of $X_1 \times \ldots \times X_{n-1}$.

Then choose a model M over which everything is defined and let p_i be an M-invariant global type in D_i, $1 \leq i \leq n$. (One may proceed as follows. Extend $D_i \cap M$ to an ultrafilter $\mathcal{U}$ on M. Then let $p_{i,\mathcal{U}} := \{\psi(\bar{x}, \bar{b}) \colon \bar{b} \in \mathfrak{M}, \psi(M, \bar{b}) \in \mathcal{U}\}$). □

Proof continued.

Let $\bar{a} := (a_1, \ldots, a_n)$ be a realization of $p_1 \otimes \ldots \otimes p_n \upharpoonright M$. We apply the preceding lemma (D_i is also quasi-minimal), so $p_1 \otimes \ldots \otimes p_n \upharpoonright M$ is broad.

Suppose that $\bar{a} \notin P$.
By choice of $D_1, \ldots, D_n$, for any $b \in D_n$,
$\{(u_1, \ldots, u_{n-1}) \in D_1 \times \ldots \times D_{n-1} : (u_1, \ldots, u_{n-1}, b) \notin P\}$ is not a broad subset of $X_1 \times \ldots \times X_{n-1}$.
Since $tp(a_1, \ldots, a_{n-1})$ is broad, we get that $\bar{a} \in P$. □

Lemma (Lemma 4.10)

Let $(X_1, \ldots, X_n, P)$ be a critical coordinate configuration and Q a quasi-minimal set. Let A be a small set of parameters over which the X_i, $1 \leq i \leq n$ and P, Q are defined.
Let $(a_1, \ldots, a_n, b) \in X_1 \times \ldots \times X_n \times Q$ such that $tp(a_1, \ldots, a_n, b/A)$ is broad and $(a_1, \ldots, a_n) \in P$.
Let $s = a_1 + \ldots + a_n + b$. Then $b \notin acl(A, s)$.

Proof.

By the way of contradiction suppose that $b \in acl(As)$. Then by Lemma 4.4, there is a broad subset $\tilde{P}$ of $X_1 \times \ldots \times X_n \times Q$ with $(a_1, \ldots, a_n, b) \in \tilde{P}$ and $(X_1, \ldots, X_n, Q, \tilde{P})$ is a coordinate configuration with $s \in \pi(\tilde{P})$. This contradicts that $(X_1, \ldots, X_n, P)$ be a critical coordinate configuration (since $rk(Q) > 0$). □

A technical result on products of types

Lemma (Lemma 4.11)

Let A be small set and p a global A-invariant type. Let $b \notin acl(A)$ and $a \models p \upharpoonright Ab$. Then there is a small model $A \subset \mathcal{M}$ and a global M-invariant type r such that $(a, b) \models p \otimes r \upharpoonright M$, namely $b \models r \upharpoonright M$ and $a \models p \upharpoonright Mb$.

Proof.

Let $b_1, b_2, \ldots$ be an A-indiscernible sequence of pairwise distinct elements realizing $tp(b/A)$. Let M_0 be a small model containing A. Let $b'_1, b'_2, \ldots$ an M_0-indiscernible sequence extracted from the first sequence. In particular b'_i realizes $tp(b/A)$ and these elements are still pairwise distinct.

Let $\sigma \in Aut(\mathfrak{M}/A)$ sending b'_1 to b and let $M_1 = \sigma(M_0)$. □

Continued.

Note that $A \subset M_1$. The sequence $\sigma(b_1'), \sigma(b_2'), \ldots,$ is a M_1-indiscernible sequence and consists of distinct elements. In particular, $b \notin acl(M_1)$. Let r_0 be a type over $\mathfrak{M}$, coheir of $tp(b/M_1)$.

(Again extend $\{\varphi(M_1, \bar{c}) : \varphi(x, \bar{c}) \in tp(b/M_1)\}$ to an ultrafilter $\mathcal{U}$ on $M_1^{|x|}\}$, then $\psi(x, \bar{d}) \in r_0$ iff $\psi(M_1, \bar{d}) \in \mathcal{U}$, $\bar{d} \in \mathfrak{M}$.) □

Continued.

Since $b \notin acl(M_1)$, r_0 has infinitely many realizations. Let a' be a realization of $p \restriction M_1 b$. Thus (a', b) realizes $p \otimes r_0 \restriction M_1$. By hypothesis, a realizes $p \restriction Ab$ and so does a'. So there is an automorphism $\tau \in Aut(\mathfrak{M}/Ab)$ such that $\tau(a') = a$. Let $M := \tau(M_1)$ and $r = \tau(r_0)$. Then (a, b) realizes $p \otimes r \restriction M$ since (a', b) realizes $p \otimes r_0 \restriction M_1$. □

Lemma (Lemma 4.12)

Let Y be a critical set of rank ρ and Q be quasi-minimal and let $t \geq 1$ an integer. There exist pairwise distinct $q_1, \ldots, q_t \in Q$ such that

$$rk(\bigcap_{i=1}^{t}(Y + q_i)) = \rho.$$

Note that the lemma implies that $\bigcap_{i=1}^{t}(Y + q_i)$ is critical. (Indeed a translate of a critical set is critical and if a subset of a critical set has the same rank then it is also critical.)

Proof (by contradiction):

So for any distinct $q_1, \ldots, q_t \in Q$, $rk(\bigcap_{i=1}^{t}(Y + q_i)) \leq \rho - 1$ (1). Let $(X_1, \ldots, X_n, P)$ be a critical configuration with target Y. By 3.23, $\rho = \sum_{i=1}^{n} rk(X_i)$. By 4.6, there exist a small model M and non-algebraic global M-invariant types p_i on X_i such that if $a \models p_1 \otimes \ldots \otimes p_n \upharpoonright M$, then $a \in P$. Furthermore we may assume that Q is defined over M and that there is a non-algebraic M-invariant type p_0 containing Q. □

Claim (4.13)

For $k \in \mathbb{N}^*$, let

$\Omega_k := \{(a_{1,1}, \ldots, a_{1,n}, \ldots, a_{k,1}, \ldots, a_{k,n}, q_0) \in (X_1 \times \ldots \times X_n)^k \times Q$ such that

1. for each $i \in [k]$, $(a_{i,1}, \ldots, a_{i,n}) \in P$,
2. there are infinitely many $q \in Q$ such that $\bigwedge_{i=1}^{k} \left((q_0 + \sum_{j=1}^{n} a_{i,j}) \in Y + q\right)$.

Then for $k >> 0$, Ω_k is not a broad subset of $(X_1 \times \ldots \times X_n)^k \times Q$.

Note that since $\exists^\infty$ is eliminated, the sets Ω_k are definable.

Proof of Claim (by contradiction).

Let $h := rk(Q) > 0$. Choose k large enough such that $t.h + k(\rho - 1) < h + k.\rho$, equivalently $h.(t-1) < k$. By 3.23, if Ω_k were broad, $rk(\Omega_k) = h + k.\rho$. In particular Ω_k would contain a tuple of that rank (over M) (2). Let $(a_{1,1}, \ldots, a_{1,n}, \ldots, a_{k,1}, \ldots, a_{k,n}, q_0)$ be such tuple. For $i \in [k]$, let $s_i := \sum_{j=1}^{n} a_{i,j}$. By definition of Ω_k, $(a_{i,1}, \ldots, a_{i,n}) \in P$. So $s_i \in Y (= \pi(P))$. □

Proof continued.

Since the fibers of π are finite, $(a_{i,1}, \dots, a_{i,n}) \in acl(s_i M)$. Again by definition of Ω_k, there are infinitely many $q \in Q$ such that $\{q_0 + s_1, \dots, q_0 + s_k\} \in Y + q$. So we may choose $q_1, \dots, q_{t-1}$ pairwise distinct and not equal to q_0 such that $q_0 + s_i \in \bigcap_{\ell=1}^{t-1} Y + q_\ell$, $i \in [k]$ (and so $q_0 + s_i \in \bigcap_{\ell=0}^{t-1} Y + q_\ell$). We have $rk(s_i/Mq_0, \dots, q_{t-1}) = rk((a_{i,1}, \dots, a_{i,n})/Mq_0, \dots, q_{t-1}) \leq rk(\bigcap_{\ell=0}^{t-1} Y + q_\ell) < \rho$ (by (1)). By subadditivity of dp-rank,

$$rk((a_{1,1}, \dots, a_{1,n}, \dots, a_{k,1}, \dots, a_{k,n}, q_0, q_1, \dots, q_{t-1})/M) \leq k(\rho - 1) + t.h,$$

contradicting (2) (recall that k has been chosen such that $k(\rho - 1) + t.h < k\rho + h$).

End of proof of the claim. □

Fix k such that $h.(t-1) < k$ and so Ω_k is not broad. Choose $(a_{1,1}, \dots, a_{1,n}, \dots, a_{k,1}, \dots, a_{k,n}, q_0)$ realizing $(p_1 \otimes \dots \otimes p_n)^{\otimes k} \otimes p_0$ over M. Let $s_i := \sum_{j=1}^{n} a_{i,j}$, $i \in [k]$. Recall that each $\bar{a}_i := (a_{i,1}, \dots, a_{i,n}) \in P$ and so $s_i \in Y$. By Lemma 4.5, $tp(\bar{a}_i, q_0)/M)$ is broad and so $(\bar{a}_i, q_0) \notin \Omega_k$. So there are only finitely many $q \in Q$ such that $\bigwedge_{i=1}^{k} \left(q_0 + s_i \in Y + q\right)$. Since $s_i \in Y$, q_0 is among these q's, which implies that $q_0 \in acl(M, s_1 + q_0, \dots, s_k + q_0)$. Choose ℓ minimal such that $q_0 \in acl(M, s_1 + q_0, \dots, s_\ell + q_0)$. Note that $\ell \geq 1$, since $tp(q_0/M) = p_0$ is non-algebraic. Let $M' := M \cup \{s_1 + q_0, \dots, s_{\ell-1} + q_0\}$. By choice of ℓ, $q_0 \notin acl(M')$; also note that $M'q_0 \subset dcl(M, q_0, (a_{i,j})_{1 \leq i \leq n, 1 \leq j \leq \ell-1})$.

We are in position to apply Lemma 4.11. Indeed, $q_0 \notin acl(M')$, $\bar{a}_\ell$ realizes the M-invariant type $p_1 \otimes \ldots \otimes p_n$ over $M'q_0$. So we can find N a small model containing M' and a N-invariant type r such that $\bar{a}_\ell q_0$ realizes $p_1 \otimes \ldots \otimes p_n \otimes r \restriction N$, namely q_0 realizes $r \restriction N$ (in particular r contains Q) and $\bar{a}_\ell$ realizes $p_1 \otimes \ldots \otimes p_n \restriction Nq_0$. By Lemma 4.5, $tp(\bar{a}_\ell, q_0/N)$ is broad. Recall that $(X_1, \ldots, X_n, P)$ was a critical coordinate configuration $\bar{a}_\ell \in P$, Q a quasi-minimal set, $(\bar{a}_\ell, q_0) \in X_1 \times \ldots, X_n \times Q$, with a broad type over N (over which everything is defined). So by Lemma 4.10, $q_0 \notin acl(s_\ell + q_0 N)$. However ℓ was chosen such that
$q_0 \in acl(M, s_1 + q_0, \ldots, s_\ell + q_0) \subset acl(M', s_\ell + q_0)$, a contradiction.