DDC reading group
Shelah's conjecture for dp finite fields
Christmas review

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Goal

The aim of this reading group is to study the proof of the following

Theorem (Johnson, 2020), [dpVI, Cor 4.16(4)]

Every infinite dp finite field is algebraically closed, real closed, or admits a nontrivial henselian valuation.

Main sources (all available on ArXiv):

dpI Will Johnson, Dp finite fields I: infinitesimals and positive characteristic, 2019.

dpII Will Johnson, Dp finite fields II: the canonical topology and its relation to henselianity, 2019.

dpIII Will Johnson, Dp finite fields III: inflators and directories, 2019.

dpIV Will Johnson, Dp finite fields IV: the rank 2 picture, 2020.

dpV Will Johnson, Dp finite fields V: topological fields of finite weight, 2020.

dpVI Will Johnson, Dp finite fields VI: the dp finite Shelah conjecture, 2020.

Dp minimal fields

Johnson's proof mimics his proof of the classification of dp-minimal fields.

Proof outline for the dp-minimal case

(a) (Existence conjecture) Construct a 'canonical topology' on a sufficiently saturated unstable dp-minimal field. Show this is a definable V-topology.

(b) (Henselianity conjecture) Show

$$(K, v) \text{ dp-minimal} \implies v \text{ henselian.}$$

(c) Wrap up using Jahnke-Koenigsmann.

Note: The hardest part is (a).

Classification of dp-minimal fields

Theorem (Johnson, [PhD thesis: Fun with Fields, Berkeley, 2015])

A field K is dp-minimal if and only if K is perfect and there exists a valuation v on K such that:

1. v is henselian.
2. v is defectless (i.e., any finite valued field extension (L, v) of (K, v) is defectless, i.e., satisfies $[L : K] = [vL : vK][Lv : Kv]$).
3. The residue field Kv is either an algebraically closed field of positive characteristic or elementarily equivalent to a local field of characteristic 0.
4. The value group vK is almost divisible, i.e., $[vK : n(vK)] < \infty$ for all n.
5. If $\mathrm{char}(Kv) = p \neq 0$ then $[-v(p), v(p)] \subseteq p(vK)$.

Steps in the dp-finite case

STEP 1:

Construct the ‘canonical topology’ on a (sufficiently saturated) unstable field $\mathbb{K}$ of finite dp-rank n.

- There is a notion of heavy sets on dp-finite fields. (Definition today!)
- Intuition: A set $X \subseteq \mathbb{K}$ is heavy if and only if $dp - rk(X) = n$
- The sets

$$X -_{\infty} X = \{\delta \in \mathbb{K} \,|\, X \cap (X + \delta) \text{ is heavy}\}$$

  form a basis of neighbourhoods of zero for a (unique) Hausdorff non-discrete group topology on $(\mathbb{K}, +)$.

$\rightsquigarrow$ Will be done in January, cf. [dpI, Sections 3-6]

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Steps in the dp-finite case

STEP 2:

The 'canonical topology' on a (sufficiently saturated) unstable field is a definable W-topology.

- This is proven in [dpV, 6.1], see Zoé's talk (19.11.20)
- So far, we only know the proof conditional on [dpI, 10.1]
- Uses the 'golden lattice' technology (recall: golden lattices of cube-rank n give rise to W_n-rings)
- Basically, [dpI, 10.1] states that taking a small submodel $K \prec \mathbb{K}$, the lattice of type-definable K-linear subspaces of $\mathbb{K}$ is a golden lattice. The right notion of smallness comes from [dpI, section 8].

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Steps in the dp-finite case

STEP 3:
Show the canonical topology is local and has a unique V-topological coarsening.

- This is proven in [dpVI, 4.15], see Sylvy's talk (19.11.20)
- Recall: Any field topology τ on K gives rise to a ring $R_\tau \subseteq K^*$, where K^* denotes an ultrapower, via

$$R_\tau = \bigcup_{B \in \tau} B^*$$

. This ring is $\vee$-definable over K.

- A W-topology is local if R_τ is a local ring.

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STEP 4:

The V-topological coarsenings of the canonical topology are exactly the definable V-topologies on K.

- This is [dpV, 6.6], see Zoé's talk (19.11.20)
- Definability of the V-topological coarsenings is [dpV, 4.10], see Zoé's earlier talk (22.10.20)
- That these are all the definable V-topologies requires [dpII, 5.10] which we will see in January/February.

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Steps in the dp-finite case - Existence conjecture

Existence Conjecture

Combining steps 1-4, we get a unique definable V-topology on a sufficiently saturated unstable dp-finite field. Thus, $\mathbb{K}$ admits an externally definable valuation ring.

- The fact that definable V-topologies give rise to externally definable valuation rings was shown in [Halevi-Hasson-Jahnke, Definable V-topologies, henselianity and NIP]

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Henselianity Conjecture

Combining steps 1-4, we get a unique definable V-topology on a sufficiently saturated unstable dp-finite field. Thus, the henselianity conjecture holds for dp-finite valued fields.

- The statement is [dpVI, Cor 4.16(3)]
- The proof was essentially covered by Zoé (19.11.20), in her proof of [dpV, 6.8].

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Steps in the dp-finite case - Wrapping up

The end

Let K be an infinite dp-finite field that is neither real closed nor algebraically closed. Then K admits a definable nontrivial henselian valuation.

- This is [dpVI, 4.16(4)], the proof that this follows from 'Existence Conjecture' and 'Henselianity Conjecture' was presented by Zoé (19.11.20)
- uses existence of a nontrivial definable valuation which induces the henselian V-topology on any nontrivially henselian valued field (i.e., [Jahnke-Koenigsmann, Uniformly defining the canonical p-henselian valuation]).

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Consequences of the dp-finite Shelah Conjecture

Halevi-Hasson: An ordered abelian group Γ has finite dp-rank in $\mathcal{L}_{\mathrm{oag}}$ if and only $\Gamma/p\Gamma$ is finite for all but finitely many primes and has finite spines.

Corollary [Halevi-Hasson-J., A conjectural classification of str. dep. fields]

A field K is dp-finite if and only if K is perfect and there exists a valuation v on K such that:

1. v is henselian.
2. v is defectless (i.e., any finite valued field extension (L, v) of (K, v) is defectless, i.e., satisfies $[L : K] = [vL : vK][Lv : Kv]$).
3. The residue field Kv is either an algebraically closed field of positive characteristic or elementarily equivalent to a local field of characteristic 0.
4. The value group vK is dp-finite.
5. If $\mathrm{char}(Kv) = p \neq 0$ then $[-v(p), v(p)] \subseteq p(vK)$.

Dp rank versus strong dependence

Corollary

A dp finite field is dp-minimal if and only if it has bounded absolute Galois group.

Question: Is there a direct proof of this?

Theorem (Halevi-Palacin)

An ordered abelian group Γ is strongly dependent (in $\mathcal{L}_{\mathrm{oag}}$) if and only $\Gamma/p\Gamma$ is finite for all but finitely many primes and has finite spines.

Thus, one gets

Theorem (Halevi-Hasson-J.)

Assume that Shelah's conjecture holds for strongly dependent fields. Then any strongly dependent field has finite dp rank.

Henselianity conjecture

Theorem (Johnson, [dpI, Theorem 2.8])

Let (K, v) be an NIP field of positive characteristic. Then v is henselian, i.e., the henselianity conjecture holds for NIP fields of positive characteristic.

Proof ingredients:

- Infinite NIP fields of characteristic $p > 0$ admit no Artin-Schreier extensions (Kaplan-Scanlon-Wagner)
- (K, v) valued field and L/K a finite normal extension. Then every extension of v to L is definable (identifying L with K^d for $d = [L : K]$)
- Let G be an NIP group. Then G^{00} exists, i.e., the intersection of all type-definable subgroups of G of bounded index is itself a subgroup of bounded index. (Shelah)

Henselianity conjecture

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