Minicourse: Lecture 3
Applying Topology to Spaces of Countable Structures

Russell Miller

Queens College & CUNY Graduate Center

DDC Program, Part I: Virtual Semester

Mathematical Sciences Research Institute
Berkeley, CA (remotely)
Autumn 2020

Plan of the Minicourse

Week 1: Specific example: subrings of $\mathbb{Q}$.
Online discussion: Thursday, Sept. 24, 11:00 PDT.

Week 2: Computability and continuity.
Online discussion: Thursday, Oct. 1, 11:00 PDT.

Week 3: Classifications of spaces of structures.
Online discussion: Thursday, Oct. 8, 11:00 PDT.

Week 4: The space of algebraic fields.
Online discussion: Thursday, Oct. 15, 11:00 PDT.

Week 5: Other related questions.
Online discussion: Thursday, Oct. 22, 11:00 PDT.

(Also watch Caleb Springer's MSRI Junior Seminar: Oct. 20, 09:00.)

The classification of the subrings of $\mathbb{Q}$

Recall our classification of $\mathfrak{R}_{\mathbb{Q}} = \{\text{all subrings of } \mathbb{Q}\}$ by the power set $\mathcal{P}(\mathbb{P})$. Each subring is of the form $\mathbb{Z}[W^{-1}]$ with $W \subseteq \mathbb{P}$; moreover, $\mathbb{Z}[W_0^{-1}] \neq \mathbb{Z}[W_1^{-1}]$ for $W_0 \neq W_1$. This is our prototype for a *classification* of a space of countable structures:

The classification of the subrings of $\mathbb{Q}$

Recall our classification of $\mathfrak{R}_{\mathbb{Q}} = \{\text{all subrings of } \mathbb{Q}\}$ by the power set $\mathcal{P}(\mathbb{P})$. Each subring is of the form $\mathbb{Z}[W^{-1}]$ with $W \subseteq \mathbb{P}$; moreover, $\mathbb{Z}[W_0^{-1}] \neq \mathbb{Z}[W_1^{-1}]$ for $W_0 \neq W_1$. This is our prototype for a *classification* of a space of countable structures:

- The bijection between $\mathcal{P}(\mathbb{P})$ and $\mathfrak{R}_{\mathbb{Q}}$ feels effective, in both directions. It's quite transparent which $W \subseteq \mathbb{P}$ corresponds to which subring.
- $\mathcal{P}(\mathbb{P})$ is a recognizable, understandable space (of the correct cardinality). So its elements are reasonable "invariants" to use in a classification. We call W the *index* of $\mathbb{Z}[W^{-1}]$.

The classification of the subrings of $\mathbb{Q}$

Recall our classification of $\mathfrak{R}_{\mathbb{Q}} = \{\text{all subrings of } \mathbb{Q}\}$ by the power set $\mathcal{P}(\mathbb{P})$. Each subring is of the form $\mathbb{Z}[W^{-1}]$ with $W \subseteq \mathbb{P}$; moreover, $\mathbb{Z}[W_0^{-1}] \neq \mathbb{Z}[W_1^{-1}]$ for $W_0 \neq W_1$. This is our prototype for a *classification* of a space of countable structures:

- The bijection between $\mathcal{P}(\mathbb{P})$ and $\mathfrak{R}_{\mathbb{Q}}$ feels effective, in both directions. It's quite transparent which $W \subseteq \mathbb{P}$ corresponds to which subring.
- $\mathcal{P}(\mathbb{P})$ is a recognizable, understandable space (of the correct cardinality). So its elements are reasonable "invariants" to use in a classification. We call W the *index* of $\mathbb{Z}[W^{-1}]$.

(One other prototype: we classify the countable models $F \models \mathbf{ACF}_0$ by their transcendence degrees $0, 1, \ldots, \omega$ over $\mathbb{Q}$.)

This is not a formal definition: these criteria are not very well defined. We will make some of them rigorous, but choosing a classification will remain a matter of taste, an art more than a science.

Creating a topological space

Let $\mathscr{S}$ be a class of (isomorphism types of) countable structures, in a signature L. Augment L with constants $\{c_n : n \in \mathbb{N}\}$, and fix a Gödel coding for atomic sentences in the larger signature.

Creating a topological space

Let $\mathscr{S}$ be a class of (isomorphism types of) countable structures, in a signature L. Augment L with constants $\{c_n : n \in \mathbb{N}\}$, and fix a Gödel coding for atomic sentences in the larger signature. We first consider

$$\mathscr{S}_{\mathbb{N}} = \{L\text{-structures } \mathcal{S} \text{ with domain } \mathbb{N} : \ [\mathcal{S}] \in \mathscr{S}\},$$

which contains copies of every isomorphism type in $\mathscr{S}$.

Creating a topological space

Let $\mathscr{S}$ be a class of (isomorphism types of) countable structures, in a signature L. Augment L with constants $\{c_n : n \in \mathbb{N}\}$, and fix a Gödel coding for atomic sentences in the larger signature. We first consider

$$\mathscr{S}_{\mathbb{N}} = \{L\text{-structures } \mathcal{S} \text{ with domain } \mathbb{N} : \ [\mathcal{S}] \in \mathscr{S}\},$$

which contains copies of every isomorphism type in $\mathscr{S}$. Let

$$\text{Codes}(\mathscr{S}_{\mathbb{N}}) = \{X \in 2^{\mathbb{N}} : X \text{ codes the atomic diagram of some } \mathcal{S} \in \mathscr{S}_{\mathbb{N}}\},$$

which is a subspace of $2^{\mathbb{N}}$, with the subspace topology.

Creating a topological space

Let $\mathscr{S}$ be a class of (isomorphism types of) countable structures, in a signature L. Augment L with constants $\{c_n : n \in \mathbb{N}\}$, and fix a Gödel coding for atomic sentences in the larger signature. We first consider

$$\mathscr{S}_{\mathbb{N}} = \{L\text{-structures } \mathcal{S} \text{ with domain } \mathbb{N} : \ [\mathcal{S}] \in \mathscr{S}\},$$

which contains copies of every isomorphism type in $\mathscr{S}$. Let

$$\text{Codes}(\mathscr{S}_{\mathbb{N}}) = \{X \in 2^{\mathbb{N}} : X \text{ codes the atomic diagram of some } \mathcal{S} \in \mathscr{S}_{\mathbb{N}}\},$$

which is a subspace of $2^{\mathbb{N}}$, with the subspace topology.
Finally, mod out by the relation of being isomorphic:

$$\text{Codes}(\mathscr{S}_{\mathbb{N}})/\cong$$

under the quotient topology. We name this space $\mathscr{S}$, forsaking the original space. (Both contain the same isomorphism types.)

Testing this out

Let's try this, on the class $(\mathfrak{R}_{\mathbb{Q}})_{\mathbb{N}}$ of all $(+,\cdot)$-structures on the domain $\mathbb{N}$, given by the Gödel codes of their atomic diagrams. $\mathfrak{R}_{\mathbb{Q}}$ itself is the quotient space $(\mathfrak{R}_{\mathbb{Q}})_{\mathbb{N}}/\cong$.

We think that the map that accepts the atomic diagram $\Delta(R)$ of a presentation R of a subring of $\mathbb{Q}$, and outputs $\{p \in \mathbb{P} : \frac{1}{p} \in R\}$, should be a continuous map from $(\mathfrak{R}_{\mathbb{Q}})_{\mathbb{N}}$ to Cantor space. (Since it is constant on $\cong$-classes, the map from $(\mathfrak{R}_{\mathbb{Q}})_{\mathbb{N}}/\cong$ to $2^{\mathbb{N}}$ will also be continuous, and should be a homeomorphism.) Indeed, it should be computable.

Testing this out

Let's try this, on the class $(\mathfrak{R}_{\mathbb{Q}})_{\mathbb{N}}$ of all $(+,\cdot)$-structures on the domain $\mathbb{N}$, given by the Gödel codes of their atomic diagrams. $\mathfrak{R}_{\mathbb{Q}}$ itself is the quotient space $(\mathfrak{R}_{\mathbb{Q}})_{\mathbb{N}}/\cong$.

We think that the map that accepts the atomic diagram $\Delta(R)$ of a presentation R of a subring of $\mathbb{Q}$, and outputs $\{p \in \mathbb{P} : \frac{1}{p} \in R\}$, should be a continuous map from $(\mathfrak{R}_{\mathbb{Q}})_{\mathbb{N}}$ to Cantor space. (Since it is constant on $\cong$-classes, the map from $(\mathfrak{R}_{\mathbb{Q}})_{\mathbb{N}}/\cong$ to $2^{\mathbb{N}}$ will also be continuous, and should be a homeomorphism.) Indeed, it should be computable.

Indeed, for a set $\mathcal{U}_{\{p\},\emptyset} \subseteq 2^{\mathbb{P}}$, the inverse image $\mathcal{V}$ is the set of all presentations that contain $\frac{1}{p}$. It only takes a finite portion of $\Delta(R)$ to show that an element $x \in R$ satisfies $x + x + \cdots + x = 1$ (or $p \cdot x = 1$). So, for every $R \in \mathcal{V}$, every other ring whose atomic diagram starts with that same finite portion will also lie in $\mathcal{V}$. Thus $\mathcal{V}$ is open.

Testing this out

Let's try this, on the class $(\mathfrak{R}_{\mathbb{Q}})_{\mathbb{N}}$ of all $(+,\cdot)$-structures on the domain $\mathbb{N}$, given by the Gödel codes of their atomic diagrams. $\mathfrak{R}_{\mathbb{Q}}$ itself is the quotient space $(\mathfrak{R}_{\mathbb{Q}})_{\mathbb{N}}/\cong$.

We think that the map that accepts the atomic diagram $\Delta(R)$ of a presentation R of a subring of $\mathbb{Q}$, and outputs $\{p \in \mathbb{P} : \frac{1}{p} \in R\}$, should be a continuous map from $(\mathfrak{R}_{\mathbb{Q}})_{\mathbb{N}}$ to Cantor space. (Since it is constant on $\cong$-classes, the map from $(\mathfrak{R}_{\mathbb{Q}})_{\mathbb{N}}/\cong$ to $2^{\mathbb{N}}$ will also be continuous, and should be a homeomorphism.) Indeed, it should be computable.

Indeed, for a set $\mathcal{U}_{\{p\},\emptyset} \subseteq 2^{\mathbb{P}}$, the inverse image $\mathcal{V}$ is the set of all presentations that contain $\frac{1}{p}$. It only takes a finite portion of $\Delta(R)$ to show that an element $x \in R$ satisfies $x + x + \cdots + x = 1$ (or $p \cdot x = 1$). So, for every $R \in \mathcal{V}$, every other ring whose atomic diagram starts with that same finite portion will also lie in $\mathcal{V}$. Thus $\mathcal{V}$ is open.

BUT WAIT.....

This doesn't work!

So inverses images of sets $\mathcal{U}_{\{p\},\emptyset}$ are open. But for inverse images of sets $\mathcal{U}_{\emptyset,\{p\}}$, the story is different!

No finite portion of the atomic diagram $\Delta(R)$ can ensure that $\frac{1}{p} \notin R$. In fact, for each n, every ring has a presentation in which $\Delta(R)\restriction n$ looks just like a presentation of $\mathbb{Z}$.

So our map is neither computable nor continuous! What went wrong?

This doesn't work!

So inverses images of sets $\mathcal{U}_{\{p\},\emptyset}$ are open. But for inverse images of sets $\mathcal{U}_{\emptyset,\{p\}}$, the story is different!

No finite portion of the atomic diagram $\Delta(R)$ can ensure that $\frac{1}{p} \notin R$. In fact, for each n, every ring has a presentation in which $\Delta(R)\restriction n$ looks just like a presentation of $\mathbb{Z}$.

So our map is neither computable nor continuous! What went wrong?

(In fact, we proved on the last slide that this is a continuous map into $2^{\mathbb{N}}$ with the Scott topology, where the sets $\mathcal{U}_{\{p\},\emptyset}$ form a subbasis. With this topology, the map turns out to be a homeomorphism. But the Scott space is not homeomorphic to Cantor space, so we are totally sunk.)

Analysis of the problem

Open sets in Cantor space are based on finitely much positive *and negative* information about elements $X \in 2^{\mathbb{N}}$. When we try to compute $W = \{p \in \mathbb{P} : \frac{1}{p} \in R\}$, we need $\Delta(R)$ to tell us which primes are $\notin W$, as well as which ones are $\in W$.

Analysis of the problem

Open sets in Cantor space are based on finitely much positive *and negative* information about elements $X \in 2^{\mathbb{N}}$. When we try to compute $W = \{p \in \mathbb{P} : \frac{1}{p} \in R\}$, we need $\Delta(R)$ to tell us which primes are $\notin W$, as well as which ones are $\in W$.

Solution: add a unary *invertibility predicate* to our signature. Now our symbols will be $+$, $\cdot$, and $\mathbb{I}$, with an axiom

$$\forall x \, [\, \mathbb{I}(x) \iff (\exists y)\, x \cdot y = 1].$$

Being definable, $\mathbb{I}$ does not change the isomorphism relation. But now, if $\frac{1}{p} \notin R$, then $\Delta(R)$ will eventually reveal an element x such that $x = 1 + 1 + \cdots + 1$ (p times) and $\neg\mathbb{I}(x)$. Any other ring whose atomic diagram begins with this same information must also omit $\frac{1}{p}$.

Problem solved

Now, with $\mathbb{I}$ in the signature, the map we defined above is computable, from $(\mathfrak{R}_{\mathbb{Q}})_{\mathbb{N}}$ onto $2^{\mathbb{N}}$, and is continuous for the Cantor topology on $2^{\mathbb{N}}$. The inverse image of $\mathcal{U}_{\emptyset,\{p\}}$ is now defined by a finite property of atomic diagrams.

Problem solved

Now, with $\mathbb{I}$ in the signature, the map we defined above is computable, from $(\mathfrak{R}_{\mathbb{Q}})_{\mathbb{N}}$ onto $2^{\mathbb{N}}$, and is continuous for the Cantor topology on $2^{\mathbb{N}}$. The inverse image of $\mathcal{U}_{\emptyset,\{p\}}$ is now defined by a finite property of atomic diagrams.

To compute the inverse map, we need to accept any $W \in 2^{\mathbb{N}}$ as an oracle, and compute the atomic diagram of a copy of the subring $\mathbb{Z}[W^{-1}]$. With the W-oracle, this is not difficult. Notice that the oracle allows us to compute $\mathbb{I}$ in this subring as well, so we can output the atomic diagram of $\mathbb{Z}[W^{-1}]$ in the expanded signature, as required.

So our computable map from $(\mathfrak{R}_{\mathbb{Q}})_{\mathbb{N}}/\cong$ to $2^{\mathbb{N}}$ has a computable inverse, and thus is a homeomorphism onto Cantor space.

Bigger picture

We had reasons for viewing Cantor space as a classification of $\{$subrings of $\mathbb{Q}\}$:

- The bijection between $\mathcal{P}(\mathbb{P})$ and $\mathfrak{R}_{\mathbb{Q}}$ feels effective, in both directions. It's quite transparent which $W \subseteq \mathbb{P}$ corresponds to which subring.
- $\mathcal{P}(\mathbb{P})$ is a recognizable, understandable space (of the correct cardinality). So its elements are reasonable "invariants" to use in a classification. We call W the *index* of $\mathbb{Z}[W^{-1}]$.

In retrospect, to make the bijection computable, we need the predicate $\mathbb{I}$. Without $\mathbb{I}$, $\{$subrings of $\mathbb{Q}\}$ is homeomorphic to the Scott space: still $2^{\mathbb{N}}$, but with a different topology. (The Scott topology is coarser than the Cantor topology; this reflects that one signature contains the other.)

Bigger picture

We had reasons for viewing Cantor space as a classification of $\{$subrings of $\mathbb{Q}\}$:

- The bijection between $\mathcal{P}(\mathbb{P})$ and $\mathfrak{R}_{\mathbb{Q}}$ feels effective, in both directions. It's quite transparent which $W \subseteq \mathbb{P}$ corresponds to which subring.
- $\mathcal{P}(\mathbb{P})$ is a recognizable, understandable space (of the correct cardinality). So its elements are reasonable "invariants" to use in a classification. We call W the *index* of $\mathbb{Z}[W^{-1}]$.

In retrospect, to make the bijection computable, we need the predicate $\mathbb{I}$. Without $\mathbb{I}$, $\{$subrings of $\mathbb{Q}\}$ is homeomorphic to the Scott space: still $2^{\mathbb{N}}$, but with a different topology. (The Scott topology is coarser than the Cantor topology; this reflects that one signature contains the other.)

Which is preferable? Depends on your tastes and your needs. Again, this is an art, not a science!

Generalizing the process

Here is our broad process for putting a topology on a class of isomorphism types of countable (infinite) structures.

Recipe

1. Consider the set $\mathscr{S}_{\mathbb{N}}$ of all structures in the class with domain $\mathbb{N}$.
2. View the atomic diagram $\Delta(M)$ of each $M \in \mathscr{S}_{\mathbb{N}}$ as an element of $2^{\mathbb{N}}$, using a fixed Gödel coding.
3. Give $\mathscr{S}_{\mathbb{N}}$ the subspace topology inherited from Cantor space $2^{\mathbb{N}}$.
4. Mod out by the relation $\cong$ of isomorphism on structures.

View the quotient space as $\mathscr{S}$ itself: it contains one $\cong$-class for each isomorphism type in the original class, and it has the quotient topology.

Generalizing the process

Here is our broad process for putting a topology on a class of isomorphism types of countable (infinite) structures.

Recipe

1. Consider the set $\mathscr{S}_{\mathbb{N}}$ of all structures in the class with domain $\mathbb{N}$.
2. View the atomic diagram $\Delta(M)$ of each $M \in \mathscr{S}_{\mathbb{N}}$ as an element of $2^{\mathbb{N}}$, using a fixed Gödel coding.
3. Give $\mathscr{S}_{\mathbb{N}}$ the subspace topology inherited from Cantor space $2^{\mathbb{N}}$.
4. Mod out by the relation $\cong$ of isomorphism on structures.

View the quotient space as $\mathscr{S}$ itself: it contains one $\cong$-class for each isomorphism type in the original class, and it has the quotient topology.

The chef may add definable predicates to the signature as desired, prior to step (2). ($L_{\omega_1\omega}$-definable predicates are acceptable.)

Test drive: algebraically closed fields

Earlier we mentioned a very simple space: countable models of $\mathbf{ACF}_0$. This is a countable space, indexed by transcendence degrees over $\mathbb{Q}$.

Test drive: algebraically closed fields

Earlier we mentioned a very simple space: countable models of $\mathbf{ACF}_0$. This is a countable space, indexed by transcendence degrees over $\mathbb{Q}$.

- In the pure signature $(+, \cdot)$ of fields, we get the indiscrete topology. Every finite piece of every ACF appears in every other ACF.

Test drive: algebraically closed fields

Earlier we mentioned a very simple space: countable models of $\mathbf{ACF}_0$. This is a countable space, indexed by transcendence degrees over $\mathbb{Q}$.

- In the pure signature $(+, \cdot)$ of fields, we get the indiscrete topology. Every finite piece of every ACF appears in every other ACF.
- Add a finitary *dependence predicate D*:

$$D(x_1, \ldots, x_n) \iff \vec{x} \text{ is algebraically dependent over } \mathbb{Q}.$$

  Now the open sets are $\{$ACFs of tr.deg. $\geq d\}$, for each $d \in \mathbb{N}$.

Test drive: algebraically closed fields

Earlier we mentioned a very simple space: countable models of $\mathbf{ACF}_0$. This is a countable space, indexed by transcendence degrees over $\mathbb{Q}$.

- In the pure signature $(+,\cdot)$ of fields, we get the indiscrete topology. Every finite piece of every ACF appears in every other ACF.
- Add a finitary *dependence predicate D*:

$$D(x_1,\ldots,x_n) \iff \vec{x} \text{ is algebraically dependent over } \mathbb{Q}.$$

  Now the open sets are $\{\text{ACFs of tr.deg.} \geq d\}$, for each $d \in \mathbb{N}$.
- Add a finitary *spanning predicate S* (but not D):

$$S(x_1,\ldots,x_n) \iff F \text{ is algebraic over } \mathbb{Q}(\vec{x}).$$

  Now the open sets are $\{\text{ACFs of tr.deg.} \leq d\}$, for each $d \in \mathbb{N}$.

Test drive: algebraically closed fields

Earlier we mentioned a very simple space: countable models of $\mathbf{ACF}_0$. This is a countable space, indexed by transcendence degrees over $\mathbb{Q}$.

- In the pure signature $(+,\cdot)$ of fields, we get the indiscrete topology. Every finite piece of every ACF appears in every other ACF.
- Add a finitary *dependence predicate D*:

  $$D(x_1,\ldots,x_n) \iff \vec{x} \text{ is algebraically dependent over } \mathbb{Q}.$$

  Now the open sets are $\{\text{ACFs of tr.deg.} \geq d\}$, for each $d \in \mathbb{N}$.
- Add a finitary *spanning predicate S* (but not D):

  $$S(x_1,\ldots,x_n) \iff F \text{ is algebraic over } \mathbb{Q}(\vec{x}).$$

  Now the open sets are $\{\text{ACFs of tr.deg.} \leq d\}$, for each $d \in \mathbb{N}$.
- Add D and S: now $\{d\}$ is open (for degrees $d \in \mathbb{N}$). The space is the one-point compactification of the discrete topology on $\mathbb{N}$.

Embeddings of structures

One nice aspect of the topology on $\mathscr{S}$ is that, for $\mathcal{A}, \mathcal{B} \in \mathscr{S}$,

$$\mathcal{A} \hookrightarrow \mathcal{B} \implies \text{every open set } \mathcal{V} \text{ containing } \mathcal{A} \text{ contains } \mathcal{B}.$$

Indeed, if $\mathcal{A} \in \mathcal{V}$, some finite tuple $\vec{a}$ from $\mathcal{A}$ witnesses its membership in $\mathcal{V}$. Under an embedding h, the tuple $(h(a_1), \ldots, h(a_n))$ has the same configuration, so $\mathcal{B} \in \mathcal{V}$ as well.

Embeddings of structures

One nice aspect of the topology on $\mathscr{S}$ is that, for $\mathcal{A}, \mathcal{B} \in \mathscr{S}$,

$$\mathcal{A} \hookrightarrow \mathcal{B} \implies \text{every open set } \mathcal{V} \text{ containing } \mathcal{A} \text{ contains } \mathcal{B}.$$

Indeed, if $\mathcal{A} \in \mathcal{V}$, some finite tuple $\vec{a}$ from $\mathcal{A}$ witnesses its membership in $\mathcal{V}$. Under an embedding h, the tuple $(h(a_1), \ldots, h(a_n))$ has the same configuration, so $\mathcal{B} \in \mathcal{V}$ as well.

The converse fails. Two nonisomorphic structures with the same Σ_1-diagram will lie in the same open sets – as with distinct ACF's in the original signature above.

Beware: changing the signature often eliminates certain embeddings! After we added the spanning predicate S (which is Π_2 as an infinitary formula), the only remaining embeddings between ACF's are the isomorphisms from an ACF onto itself.

Another test: finite-branching trees

Here we consider finite-branching trees of countable (infinite) height. The signature has a predecessor function p and a constant r for the root. For $x \neq r$, $p(x)$ is the immediate predecessor of x in the tree. (For simplicity, define $p(r) = r$.)

Another test: finite-branching trees

Here we consider finite-branching trees of countable (infinite) height. The signature has a predecessor function p and a constant r for the root. For $x \neq r$, $p(x)$ is the immediate predecessor of x in the tree. (For simplicity, define $p(r) = r$.)

In this signature, for each finite tree S_0, the space has a basic open set

$$\{\text{trees } T : S_0 \text{ embeds into } T\}.$$

This space is not readily recognizable. It has the property of being a *spectral space*, which means that it is homeomorphic to the Zariski topology on the spectrum of prime ideals in some commutative ring. We don't know what ring it is, and this does not seem helpful. We need a better classification.

Adding predicates

For finite-branching trees, the usual solution is to add unary *branching predicates*. Define, for each $n \geq 0$,

$$B_n(x) \iff x \text{ has exactly } n \text{ immediate successors.}$$

Now we will define a homeomorphism H from this class onto Baire space $\mathbb{N}^{\mathbb{N}} = \{\text{countable sequences from } \mathbb{N}\}$. We will use a computable list $S_0, S_1, \ldots$ of all finite trees.

Adding predicates

For finite-branching trees, the usual solution is to add unary *branching predicates*. Define, for each $n \geq 0$,

$$B_n(x) \iff x \text{ has exactly } n \text{ immediate successors.}$$

Now we will define a homeomorphism H from this class onto Baire space $\mathbb{N}^{\mathbb{N}} = \{\text{countable sequences from } \mathbb{N}\}$. We will use a computable list $S_0, S_1, \ldots$ of all finite trees.

Given a tree T, the predicates $B_1(r), B_2(r), \ldots$ eventually tell us the number n_1 of nodes at level 1. $H(T)$ begins with $(n_1 - 1)$.

The initial segment $(n_1, \ldots, n_{k-1})$ of $H(T)$ describes the isomorphism type $T{\restriction}k$ of the first $k-1$ levels of T. With this information, we go through our list of finite trees: find the unique j with $T{\restriction}(k+1) \cong S_j$, and let

$$n_k = |\{i < j : \text{ht}(S_i) = k \;\&\; T{\restriction}k \cong S_i{\restriction}k\}|.$$

A classification of trees

The foregoing procedure was effective (using the B_n predicates), and clearly respects isomorphism, so H is a continuous map from the space into $\mathbb{N}^{\mathbb{N}}$. Moreover, it is not difficult to compute the inverse map: given $(n_1, n_2, \ldots) \in \mathbb{N}^{\mathbb{N}}$, create a root r with $n_1 + 1$ immediate successors, then use n_2 to figure out the isomorphism type up to level 2 and build that (defining $B_n(x)$ for each x at level 1), and so on. This is clearly a computable inverse to H, so our space of trees is classified by Baire space.

Possible objections

1. What does an element of $\mathbb{N}^{\mathbb{N}}$ actually tell me about a tree? It's not as intuitive as the classification of subrings of $\mathbb{Q}$ was.

Possible objections

1. What does an element of $\mathbb{N}^{\mathbb{N}}$ actually tell me about a tree? It's not as intuitive as the classification of subrings of $\mathbb{Q}$ was.
2. You required that these trees have infinite height. Why leave out the finite trees?

Possible objections

1. What does an element of $\mathbb{N}^{\mathbb{N}}$ actually tell me about a tree? It's not as intuitive as the classification of subrings of $\mathbb{Q}$ was.
2. You required that these trees have infinite height. Why leave out the finite trees?
3. I like Cantor space better. Couldn't we have classified these trees using Cantor space instead?

Possible objections

1. What does an element of $\mathbb{N}^{\mathbb{N}}$ actually tell me about a tree? It's not as intuitive as the classification of subrings of $\mathbb{Q}$ was.
2. You required that these trees have infinite height. Why leave out the finite trees?
3. I like Cantor space better. Couldn't we have classified these trees using Cantor space instead?

(3) is addressed quickly: we have shown that the space, with our predicates B_n, is not compact. Adding further predicates will keep it noncompact, because taking reducts down to this signature will be a continuous map onto this noncompact space.

For (1), this is all based on our fixed computable listing of the finite trees. If you use a good intuitive listing – or think of each n_i as a finite tree, rather than a number – this should seem reasonable.

Including finite trees

So far all structures have had domain $\mathbb{N}$. There are ways to allow the atomic diagram to enumerate its own domain instead, which allows the space to include finite structures. (This would be most important when considering algebraic field extensions of $\mathbb{Z}/(p)$ for primes p.)

Including finite trees

So far all structures have had domain $\mathbb{N}$. There are ways to allow the atomic diagram to enumerate its own domain instead, which allows the space to include finite structures. (This would be most important when considering algebraic field extensions of $\mathbb{Z}/(p)$ for primes p.)

An intriguing note: Baire space $\mathbb{N}^{\mathbb{N}}$ is homeomorphic to the space of all irrational real numbers. (Map a sequence $\vec{n}$ to the continued fraction $\frac{1}{1+n_0+\frac{1}{1+n_1+\cdots}} \in (0,1)$.) It's natural to wonder whether, if we include finite trees in our space of finite-branching trees, they might correspond to rational numbers and make the whole space homeomorphic to $\mathbb{R}$.

The immediate answer is that this fails: with the B_n predicates, each finite tree is an isolated point in the space. So one would need to play with the signature, in a natural way, to see if some version is homeomorphic to $\mathbb{R}$. It does not seem easy.

Finite-valence graphs

The classification of undirected graphs of finite valence (a.k.a. finite degree) is very similar to that of finite-branching trees, if we restrict to connected, pointed graphs (G, c).
(*Pointed* means that there is a constant c, naming some node in G. But adding an undefined constant already messes with isomorphisms.)

We add *valence predicates* V_n, analogous to the B_n's, and assume G is infinite. The space $\{$pointed connected finite-valence graphs$\}$ with the V_n's, is homeomorphic to $\mathbb{N}^{\mathbb{N}}$, using a fixed computable list of all finite pointed graphs.

Finite-valence graphs

The classification of undirected graphs of finite valence (a.k.a. finite degree) is very similar to that of finite-branching trees, if we restrict to connected, pointed graphs (G, c).
(*Pointed* means that there is a constant c, naming some node in G. But adding an undefined constant already messes with isomorphisms.)

We add *valence predicates* V_n, analogous to the B_n's, and assume G is infinite. The space $\{$pointed connected finite-valence graphs$\}$ with the V_n's, is homeomorphic to $\mathbb{N}^{\mathbb{N}}$, using a fixed computable list of all finite pointed graphs.

It's not too hard to imagine dropping the connectedness, if each connected component is pointed. But how to drop the pointedness?

Finite-valence connected (non-pointed) graphs

Two connected graphs G_0 and G_1 are isomorphic iff (with $c_0 \in G_0$ fixed) there exists $c_1 \in G_1$ s.t. $(G_0, c_0) \cong (G_1, c_1)$.

For each $c_1 \in G_1$, we can compute the index $H(G_1, c_1) \in \mathbb{N}^{\mathbb{N}}$, but it is not decidable whether this index equals $H(G_0, c_0)$. Equality in $\mathbb{N}^{\mathbb{N}}$ is a Π_1 property, and so (with V_n's but no c) it is Σ_2 whether $G_0 \cong G_1$:

$$(\exists c_1 \in G_1)\ [(G_0, c_0) \cong (G_1, c_1)].$$

Isomorphism problems

To determine whether two structures are isomorphic, it is often necessary to use jumps.

Definition

For a class $\mathscr{S}_{\mathbb{N}}$ of structures on $\mathbb{N}$, the *isomorphism problem* is the set $\{(\mathcal{A}, \mathcal{B}) \in (\mathscr{S}_{\mathbb{N}})^2 : \mathcal{A} \cong \mathcal{B}\}$. This problem is:

- Σ^0_{n+1} if $\exists$ a program Φ such that, for all $\mathcal{A}, \mathcal{B} \in \mathscr{S}_{\mathbb{N}}$,

$$\mathcal{A} \cong \mathcal{B} \iff \Phi^{(\Delta(\mathcal{A}) \oplus \Delta(\mathcal{B}))^{(n)}}(0) \text{ halts.}$$

- Π^0_{n+1} if $\exists$ a program Φ such that, for all $\mathcal{A}, \mathcal{B} \in \mathscr{S}_{\mathbb{N}}$,

$$\mathcal{A} \cong \mathcal{B} \iff \Phi^{(\Delta(\mathcal{A}) \oplus \Delta(\mathcal{B}))^{(n)}}(0) \text{ never halts.}$$

For a classification H, of course, $\mathcal{A} \cong \mathcal{B} \iff H(\Delta(\mathcal{A})) = H(\Delta(\mathcal{B}))$.

Isomorphism problems and predicates

The isomorphism problem for $(\mathfrak{R}_{\mathbb{Q}})_{\mathbb{N}}$, in the pure signature $(+, \cdot)$, is Π^0_2:

$$\mathcal{A} \cong \mathcal{B} \iff (\forall p \in \mathbb{P}) \left[((\exists x \in \mathcal{A}) x \cdot p = 1) \iff ((\exists x \in \mathcal{B}) x \cdot p = 1) \right].$$

Adding invertibility reduces this to Π^0_1, which helps explain why $\mathbb{I}$ is necessary to get a classification by Cantor space. Every space classified by Cantor space (or Baire space) must have a Π^0_1 isomorphism problem.

Isomorphism problems and predicates

The isomorphism problem for $(\mathfrak{R}_{\mathbb{Q}})_{\mathbb{N}}$, in the pure signature $(+, \cdot)$, is Π^0_2:

$$\mathcal{A} \cong \mathcal{B} \iff (\forall p \in \mathbb{P}) \, [((\exists x \in \mathcal{A}) x \cdot p = 1) \iff ((\exists x \in \mathcal{B}) x \cdot p = 1)].$$

Adding invertibility reduces this to Π^0_1, which helps explain why $\mathbb{I}$ is necessary to get a classification by Cantor space. Every space classified by Cantor space (or Baire space) must have a Π^0_1 isomorphism problem.

But the isomorphism problem for connected graphs is Σ^0_3, and becomes (properly) Σ^0_2 in the signature with the V_n's. This makes it very hard to see any definable predicates that would allow classification by a space where equality is Π^0_1. The best approach is to determine the natural equivalence relation E on Baire space for which $\mathbb{N}^{\mathbb{N}}/E$ classifies $\{$connected graphs on $\mathbb{N}\}$. This E should be Σ^0_2, to match the isomorphism problem.

Connections between classes

If $\mathscr{S}_{\mathbb{N}}$ and $\mathfrak{T}_{\mathbb{N}}$ are two classes of structures on $\mathbb{N}$, one naturally asks whether $\mathscr{S}_{\mathbb{N}}/\cong$ can be mapped computably into $\mathfrak{T}_{\mathbb{N}}/\cong$ via a map that is injective (on $\cong$-classes). If so, then the isomorphism problem for $\mathscr{S}_{\mathbb{N}}$ reduces to that for $\mathfrak{T}_{\mathbb{N}}$.

The maps in question were considered by the logic group at Notre Dame, and are known as *Turing-computable embeddings*.

Using the ideas of this lecture, it would be natural to consider cases where there is no Turing-computable embedding, and to ask which definable predicates (if any) can be added to the signature to enable such an embedding to be computed.