Minicourse: Lecture 5
Applying Topology to Spaces of Countable Structures

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DDC Program, Part I: Virtual Semester

Mathematical Sciences Research Institute
Berkeley, CA (remotely)
Autumn 2020

Plan of the Minicourse

Week 1: Specific example: subrings of $\mathbb{Q}$.
Online discussion: Thursday, Sept. 24, 11:00 PDT.

Week 2: Computability and continuity.
Online discussion: Thursday, Oct. 1, 11:00 PDT.

Week 3: Classifications of spaces of structures.
Online discussion: Thursday, Oct. 8, 11:00 PDT.

Week 4: The space of algebraic fields.
Online discussion: Thursday, Oct. 15, 11:00 PDT.

Week 5: Other related questions.
Online discussion: Thursday, Oct. 22, 11:00 PDT.

(Also watch Caleb Springer's MSRI Junior Seminar: Oct. 20, 09:00.)

The topology of $\mathfrak{Alg}_{\mathbb{Q}}$

Recall the topology on $\mathfrak{Alg}_{\mathbb{Q}}$, the space of (isomorphism types of) algebraic field extensions of $\mathbb{Q}$:

For each number field F and each $h \in \mathbb{Q}[X]$ with no roots in F, let

$$\mathcal{U}_{F,h} = \{\text{algebraic fields } K \supseteq \mathbb{Q} : F \hookrightarrow K \ \& \ h \text{ has no roots in } K\}.$$

So F is the "positive" information (essentially finite, since F is a number field) about K, and h is the "negative" information. Each such $\mathcal{U}_{F,h}$ is open, and (F, h) is called a *condition*.

In this topology, $\mathfrak{Alg}_{\mathbb{Q}}$ is homeomorphic to Cantor space, so we have the notions of Baire category on $\mathfrak{Alg}_{\mathbb{Q}}$.

Conditions

Conditions (F, h) are partially ordered by $\preceq$, where $(F, h) \preceq (F', h')$ iff

$$F \hookrightarrow F' \ \& \ (\forall r \in \overline{\mathbb{Q}})[h(r) = 0 \implies h' \text{ has a root in } F'(r)].$$

By Kronecker's Theorem, $\preceq$ is decidable. A set D of conditions is *dense* if, for every condition (F, h), D contains some $(F', h') \succeq (F, h)$.

Generic fields

A field $K \in \mathfrak{Alg}_{\mathbb{Q}}$ is *generic* if, for every arithmetical dense set D of conditions, there exists some $(F, h) \in D$ such that $K \in \mathcal{U}_{F,h}$. (That is, F embeds into K, and K contains no roots of h.)

To build a generic field K, make a list of the arithmetical dense sets $D_0, D_1, \ldots$. Start with any $(F_0, h_0) \in D_0$, then find $(F_1, h_1) \succeq (F_0, h_0)$ in D_1 (which is dense), and so on. (This is *not* a computable process.) Set $K = \cup_n F_n$, under the embeddings $F_n \hookrightarrow F_{n+1}$.

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Lemma

The generic fields form a comeager subset of $\mathfrak{Alg}_{\mathbb{Q}}$.

Each non-generic field is ruled out by some D_n, and the set of fields ruled out by a given D_n is nowhere dense. Each (F, h) has an $(F', h') \succeq (F, h)$ in D_n, and every $K \in \mathcal{U}_{F',h'}$ meets D_n.

Forcing

Definition

For a polynomial $f \in \mathbb{Q}[Y_1, \ldots, Y_k]$, a condition (F, h) *forces* the sentence $\forall \vec{y}\ f(\vec{y}) \neq 0$ if
for every $(F', h') \succeq (F, h)$ and every $\vec{y} \in (F')^k$, $f(\vec{y}) \neq 0$.

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We write $(F, h) \Vdash \forall \vec{y}\ f(\vec{y}) \neq 0$ or $(F, h) \Vdash \exists \vec{y}\ f(\vec{y}) = 0$, accordingly.

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Lemma

If K is generic and $K \in \mathcal{U}_{F,h}$, and $(F, h) \Vdash \psi$, then $K \models \psi$.

Generics force everything

Lemma

For every generic field K and every Σ_1 or Π_1 sentence ψ, there exists some condition (F, h) with $K \in \mathcal{U}_{F,h}$ such that

$$(F, h) \Vdash \psi \quad \text{or} \quad (F, h) \Vdash \neg\psi.$$

And then $K \models \psi$ or $K \models \neg\psi$ accordingly, since K is generic.

Key theorem

Theorem (Eisenträger, M., Springer, & Westrick, 2020)

It is decidable whether $(F, h) \Vdash \exists \vec{y}\ f(\vec{y}) = 0$, and also whether $(F, h) \Vdash \forall \vec{y}\ f(\vec{y}) \neq 0$.

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Corollary

For every generic $K \in \mathfrak{Alg}_{\mathbb{Q}}$, $\text{HTP}(K) \equiv_T I_K$. That is, deciding whether K contains solutions to polynomial equations in several variables is exactly as hard as deciding it for polynomial equations in one variable.

Since generics are comeager, this holds for comeager-many K.

If K is generic, eventually there is some (F, h) with $K \in \mathcal{U}_{F,h}$ for which (F, h) either forces a solution or forces no solution. Given I_K, search until we find such an (F, h), recognizing it using the theorem.

More corollaries of the EMSW theorem

Corollary

For every generic $K \in \mathfrak{Alg}_{\mathbb{Q}}$ and every presentation L of K, $\text{HTP}(L) \equiv_T R_L$, and both are low relative to $\Delta(L)$.

Again, since generics are comeager, this holds for comeager-many K.

Corollary

For every generic $K \in \mathfrak{Alg}_{\mathbb{Q}}$, there exists some presentation L of K for which $\text{HTP}(L) \equiv_T R_L$ and neither is computable from $\Delta(L)$.

Again, this holds for comeager-many K.

These follow from theorems seen at the end of Week 4.

Other spaces of structures

Question: what topology results on $\mathfrak{Alg}_{\mathbb{Q}}$ if we do not include root predicates in the signature? Partial answer: a spectral space.

Definition

A topological space T is a *spectral space* if:

- T is (quasi)-compact and T_0 ($\forall x \neq y$ some open set contains exactly one of x and y);
- The class $K^c(T)$ of all compact open subsets of T forms a basis;
- $K^c(T)$ is closed under finite intersections; and
- Every nonempty irreducible closed subset of T is the closure of a singleton.

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Theorem (Hochster, 1968)

A topological space is spectral iff it is homeomorphic to the spectrum of prime ideals of some commutative ring, under the Zariski topology.

Torsion-free abelian groups

The class $\mathfrak{TFab}_1$ of all torsion-free abelian groups of rank 1 (equivalently, subgroups of $(\mathbb{Q}, +, 0)$) shares many computable-model-theoretic properties with the class of algebraic fields. The isomorphism problem for $\mathfrak{TFab}_1$ is Σ^0_3.

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With no additional predicates, $\mathfrak{TFab}_1$ has the indiscrete topology. It is natural to add unary *divisibility predicates* for primes p:

$$D_p(x) \iff (\exists y)\, x = y + y + \cdots + y \ \ (p \text{ times}).$$

One might also want *infinitary divisibility predicates*:

$$D_{p^\infty}(x) \iff (\forall n)(\exists y)\, x = y + y + \cdots + y \ \ (p^n \text{ times}).$$

Questions: what predicates yield a reasonably natural topology on $\mathfrak{TFab}_1$? And what about $\mathfrak{TFab}_d$, the subgroups of $(\mathbb{Q}^d, +, 0)$?

Equivalence structures

An *equivalence structure* is nothing more than an equivalence relation E on the domain $\mathbb{N}$. However, the isomorphism problem for equivalence structures is Π^0_4, harder than other classes we have considered so far. (It becomes simpler for subclasses where one specifies the number of infinite equivalence classes in the structure.) What are the natural predicates here? And what topologies can result?

Differential fields

For fields, $\mathfrak{Alg}_{\mathbb{Q}}$ is both the class of all algebraic field extensions of $\mathbb{Q}$, and the class of all subfields of $\overline{\mathbb{Q}}$.

If we consider the differential closure $\widetilde{\mathbb{Q}}$ of the constant differential field $\mathbb{Q}$, then the subfields of $\widetilde{\mathbb{Q}}$ form a class distinct from the class of differentially algebraic differential field extensions of $\mathbb{Q}$. We ask about the topology of each of these two classes. Probably one should consider a root predicate for differential polynomials in one variable?

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Here we might get noncomputable homeomorphisms. The set of *constrained pairs* of differential polynomials over $\mathbb{Q}$ is Π_1, but its decidability status is unknown. It is analogous to the set of polynomials irreducible in $\mathbb{Q}[X]$: these are the generators of the principal types for fields, while constrained pairs are the generators of the principal types for differential fields.

Example of Bazhenov

Nikolay Bazhenov produced a class $\mathfrak{C}$ for which the topology is homeomorphic to the order topology on the real line $\mathbb{R}$.

Let the signature have unary predicates $P_{a,b}$, for each $a < b$ in $\mathbb{Q}$. Assume the axioms

- (for $a_1 \leq a_2 < b_2 \leq b_1$): $\forall x\ [P_{a_2,b_2}(x) \implies P_{a_1,b_1}(x)]$.
- (for $a_1 < b_1 \leq a_2 < b_2$): $\forall x\ \neg[P_{a_1,b_1}(x)\ \&\ P_{a_2,b_2}(x)]$.
- (for $a < a_1 < b_1 < b$): $\forall x\ [P_{a,b}(x) \implies (P_{a,b_1}(x) \text{ or } P_{a_1,b}(x))]$, and also $\forall x\ [(P_{a.b_1}(x)\ \&\ P_{a_1,b}(x)) \implies P_{a_1,b_1}(x)]$.

We consider the class of structures where the domain is a singleton $\{x\}$ and some $P_{a,b}(x)$ holds. Intuitively, each structure's x defines a real number r with

$$\{r\} = \cap_{P_{a,b}(x)}\ (a, b),$$

and two structures are isomorphic iff they define the same r.