Volume classes for dense actions of discrete groups

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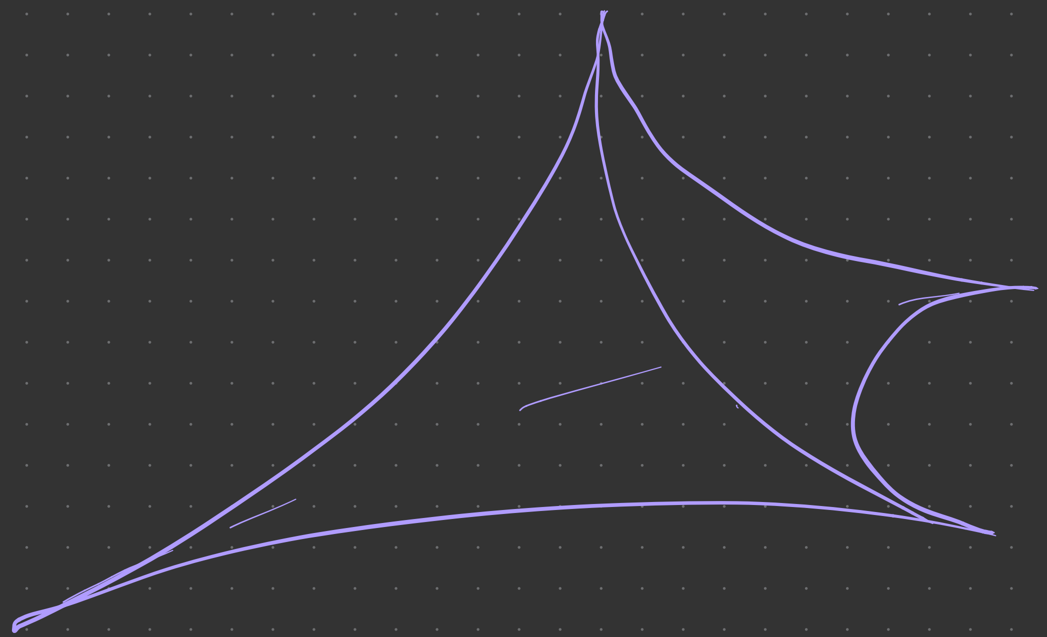


Setting

Γ - discrete gp. (f.g).

X - symm. space of non-compact type, $\dim(X) = n$.

G - $\mathrm{Isom}^+(X)$

$$\mathcal{X}(\Gamma, G) = \mathrm{Hom}(\Gamma, G) /\!/ G = \left\{ \begin{array}{c} \text{isometric actions} \\ \Gamma \curvearrowright X \end{array} \right\} / \sim .$$

Goal

Define a map $[\rho] \in \mathcal{X}(\Gamma, G) \longmapsto [\rho^* \mathrm{vol}_X] \in H^n_b(\Gamma; \mathbb{R})$

& 1) Identify interesting subsets $\subset \mathcal{X}(\Gamma, G)$ via numerical invariants.

2) " geom / top. constraints imposed

$[\rho^* \mathrm{vol}_X] \neq 0$.

Normed chain complexes

$$\Sigma_k(\Gamma) = \{k \text{ simplices} \subset \Gamma\} = \Gamma^{k+1}/\Gamma \ni [\gamma_0, \dots, \gamma_k].$$

$C_k(\Gamma;\mathbb{R}) = \langle \Sigma_k(\Gamma) \rangle_{\mathbb{R}}$, has an ℓ^1-norm:

$$\left\| \sum a_\sigma \cdot \sigma \right\|_1 = \sum |a_\sigma| \in \mathbb{R}.$$

$$(C_\bullet(\Gamma;\mathbb{R}), \partial, \|\cdot\|_1) \xmapsto[\text{dual}]{\text{topological}} (C_b^\bullet(\Gamma;\mathbb{R}), \delta, \|\cdot\|_\infty).$$

$$C_b^k(\Gamma;\mathbb{R}) = \{\text{bounded } \Gamma\text{-invnt } \Gamma^{k+1} \longrightarrow \mathbb{R}\}.$$

(Semi)-normed (co)-homology

<u>Defn</u>: $H^\bullet_b(T;\mathbb{R}) = \text{ker}s \,/\, \text{im}s = \dfrac{\{\text{bdd cocycles}\}}{\{\text{bdd co-boundaries}\}}$

$H^\bullet_b(T;\mathbb{R})$ inherits a quotient semi-norm.

$$\|\alpha\| = \inf_{a \in \alpha} \|a\|_\infty \geq 0$$

($H_\bullet(T;\mathbb{R})$ gets a semi-norm too).

`Basic' facts

1) $\rho : G \to H \rightsquigarrow \rho^* : H_b^\bullet(H) \to H_b^\bullet(G)$

semi-norm non-increasing.

2) $H_b^\bullet(M; \mathbb{R}) \cong H_b^\bullet(\pi_1 M; \mathbb{R})$ (Gromov). , M - reasonable top. space.

↖ need not be aspherical.

3) $\dim_{\mathbb{R}} H_b^k(F_2; \mathbb{R}) = \begin{cases} 0 & k=1 \\ \# \mathbb{R} & k=2, 3 \\ ?? & k \geq 4 \end{cases}$

Brooks 80's ↙ (k = 2)

← Soma 90's (k = 3)

Volume classes

Theorem (Thurston, Bucher, Lafont-Schmidt)

There is a distinguished, non-zero bounded class $[\mathrm{vol}_X] \in H^n_b(G;\mathbb{R})$.

For $X = \mathbb{H}^3$, $G = PSL_2\mathbb{C}$; fix $x \in \mathbb{H}^3$.

$\mathrm{vol}_{\mathbb{H}^3} : G^4 \longrightarrow \mathbb{R}$

$(g_0, \ldots, g_3) \longmapsto \pm$ volume of convex hull of $g_0 \cdot x, \ldots, g_3 x$

__Rmk:__ Usually "straight simplices" & volume bounds not usually easy to construct.

Fact: $\|[\mathrm{vol}_{\mathbb{H}^3}]\| = V_3 =$ volume of regular ideal Δ_3

Theorem (Soma '90s).

Γ - f.g. disc. gp. (torsion free).

$\rho: \Gamma \to PSL_2\mathbb{C}$ discrete & faithful

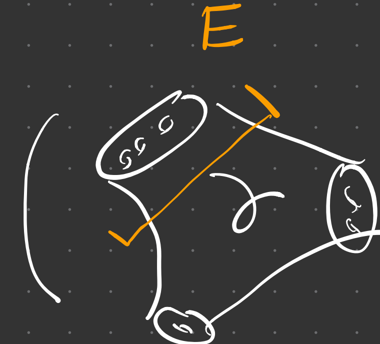
$\rho(\Gamma)$ - has ∞-co-volume

Then TFAE:

- $0 \neq [\rho^*\mathrm{vol}_{\mathbb{H}^3}] \in H^3_b(\Gamma; \mathbb{R})$
- $\|[\rho^*\mathrm{vol}_{\mathbb{H}^3}]\| = V_3$
- $\mathbb{H}^3/\rho(\Gamma)$ has a simply degenerate (relative) end.

Ends of hyperbolic 3-manifolds

$\rho : \Gamma \to PSL_2\mathbb{C}$ is as above

then $\mathbb{H}^3/_{\rho(\Gamma)} \cong \mathrm{int}\big($ 

(Thurston, Canary).

<u>Defn:</u> A simply degenerate end $E \cong S \times [0,\infty)$.
$\exists$ "hyp surfaces" $X_k \sim S \times \{0\} \subset E$ & st $\{X_k\}$ exits all compact sets:

Theorem (F.): Γ - disc. gp., $\rho_1, \rho_2 : \Gamma \to PSL_2\mathbb{C}$

If $\overline{\rho_1(\Gamma)} = PSL_2\mathbb{C}$, then

$$\|[\rho_1^* vol_{\mathbb{H}^3}]\| = V_3.$$

If $\exists\, H \leq \Gamma$ st $\overline{\rho_1(H)} = PSL_2\mathbb{C}$ but $\rho_2(H)$ is discrete or stab. a pt. or plane, then $*$

$$2V_3 \geq \|[\rho_1^* vol_{\mathbb{H}^3}] - [\rho_2^* vol_{\mathbb{H}^3}]\| \geq V_3$$

are achieved.

$(*)$ Generalizes to arbitrary collections $\{\rho_\alpha\}$

Cor: 1) $\underbrace{\langle [\rho^* \mathrm{vol}_{\mathbb{H}^3}] \mid [\rho] \in X(F_2, PSL_2\mathbb{C}),\ \rho\text{-dense} \rangle_{\mathbb{R}}}_{\dim \,=\, \#\mathbb{R}}.$

2) $\dim_{\mathbb{R}} H_b^3(PSL_2\mathbb{R};\mathbb{R}) = \#\mathbb{R}.$ (constructing "wild" $\sigma: \mathbb{C} \to \mathbb{C}$; $\overline{\sigma(\mathbb{R})} = \mathbb{C}$)

3) (Bucher-Monod): $\Gamma = SL_2(\mathbb{Z}[p_1^{-1}, \dots, p_k^{-1}])$ p_i - primes.
does not admit $\rho: \Gamma \to PSL_2\mathbb{C}$, $k \geq 1$.

Generally: • X - even dim. $n \geq 4$.

• $\Gamma \in \{$ free, surface, hyp. 3-mnfld, ... $\}$.

• $\rho : \Gamma \to G$ discrete & faithful.

(Bowen)

$$\Rightarrow \text{cheeger}\left(X/\rho(\Gamma)\right) = \inf_{M \subset X/\rho(\Gamma)} \frac{\text{Area}(\partial M)}{\text{vol}(M)} >> 0.$$

If X - rank 1 $\overset{\text{(Kim-Kim)}}{\Longrightarrow}$ $[\rho^* \text{vol}_X] = 0 \Longleftrightarrow \text{cheeger}(X/\rho) > 0$

Ingredients: • $L^{(2)}$ betti numbers

• B-S convergence

• top of uniform lattice $\leq G$.

Sketchy Idea of proof of Thm (ρ is dense $\Rightarrow [\rho^* \mathrm{vol}_{\mathbb{H}^3}] \neq 0$)

- $\exists \langle a, b\rangle \leq PSL_2\mathbb{C}$ discrete, & $M = \mathbb{H}^3 / \langle a,b\rangle$.

- $M \setminus X_k$ - compact component. called V_k.

$\mathrm{vol}(V_k) \nearrow \infty$, $\mathrm{area}(\partial V_k) \approx \mathrm{const}$

- ρ is dense $\rightsquigarrow$ $x_k, y_k \in \Gamma$ s.t. $\rho(x_k) \longrightarrow a$
$\rho(y_k) \longrightarrow b$.

$\rightsquigarrow U_k \subset \mathbb{H}^3$, "$U_k / \langle \rho(x_k), \rho(y_k)\rangle$" $\approx V_k$.

$$\underbrace{U_k}_{\tilde{V}_k}$$

- "$\mathrm{cheeger}(\rho) = 0 \Rightarrow [\rho^* \mathrm{vol}_{\mathbb{H}^3}] \neq 0$"

$\square$.

Defn: $[\alpha] \in H_n(\Gamma;\mathbb{R})$ is ε-freely approximated if $\exists$ $\varphi : F_k \to \Gamma$ and $Z \in C_n(\Gamma;\mathbb{R})$ s.t.

$$\varphi_* Z \in \alpha \quad \& \quad \|\partial Z\|_1 < \varepsilon.$$

Prop: Assume $\exists$ $\varepsilon > 0$ & $\Gamma_1, \Gamma_2, \ldots \leq G$ uniform lattice & $[\Gamma_i] = [X/\Gamma_i]$ is ε-freely approx. $\forall i$, $\mathrm{vol}(X/\Gamma_i) \nearrow \infty$.

If $\rho : F_2 \to G$ is dense,

then $[\rho^* \mathrm{vol}_X] \neq 0 \in H^n_b(F_2;\mathbb{R})$.

Rmk: Many sequences of hyp. 2, 3 mnflds are K-freely approximated for some K.