Anosov representations and counting in some PSO(p,q)-symmetric spaces

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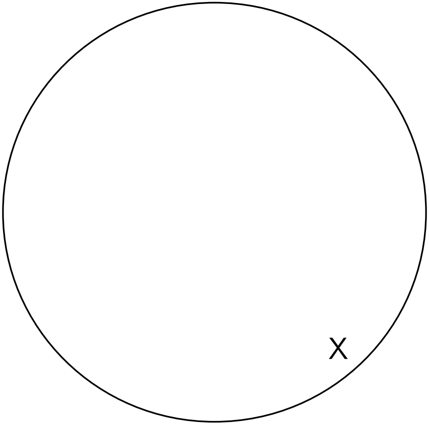

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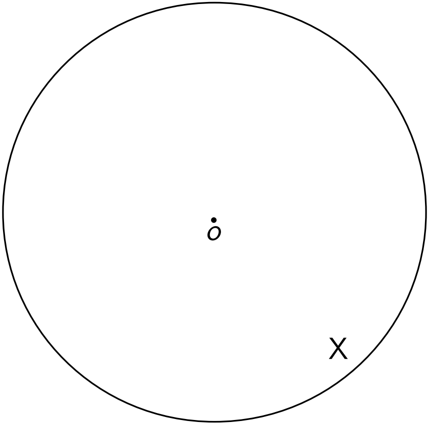

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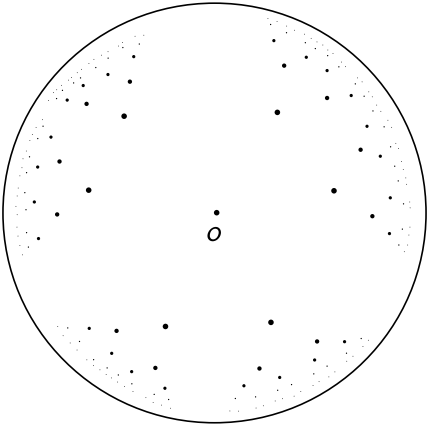

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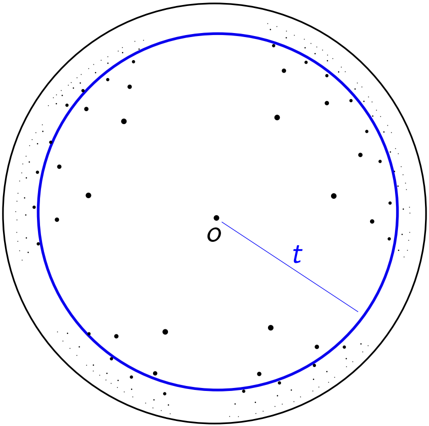

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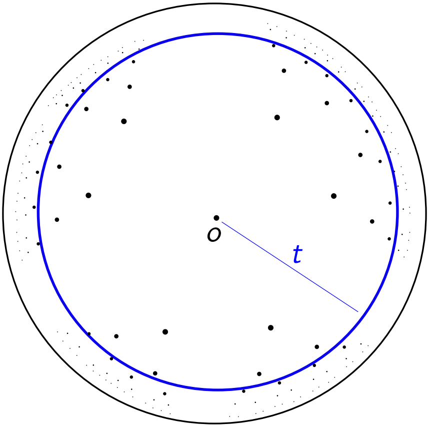

The orbital counting problem

$$\text{As } t \to \infty, \#\{g \in \Xi : \quad d_{\mathsf{X}}(o, g \cdot o) \leq t\} \sim ?.$$

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Discuss counting problems when:

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- The basepoint o is replaced by a geodesic submanifold S.
- The space X is replaced by a pseudo-Riemannian (symmetric) space.

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Discuss the link between the two countings.

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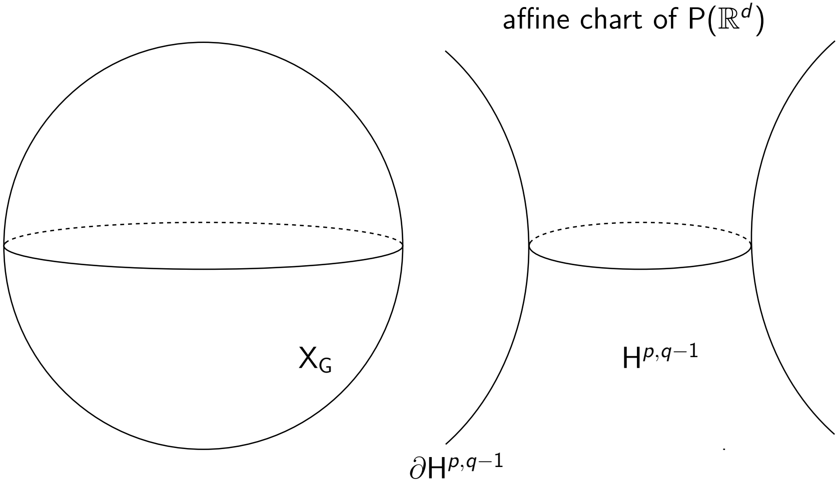
- Killing metric is proportional to $\langle \cdot, \cdot \rangle_{p,q}$.



Natural boundary of $H^{p,q-1}$: space of isotropic lines.

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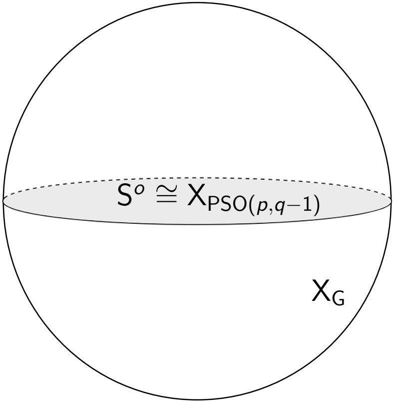

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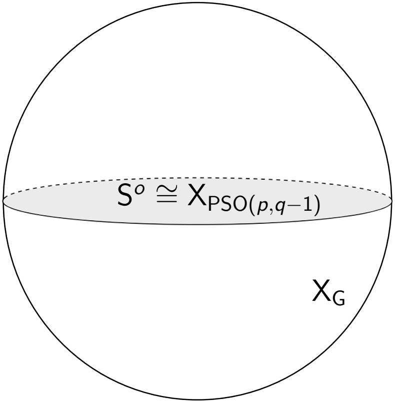

Given $o \in \mathrm{H}^{p,q-1}$, set $\mathrm{S}^o := \{\tau \in \mathrm{X}_\mathrm{G} : \quad o \subset \tau\}$.

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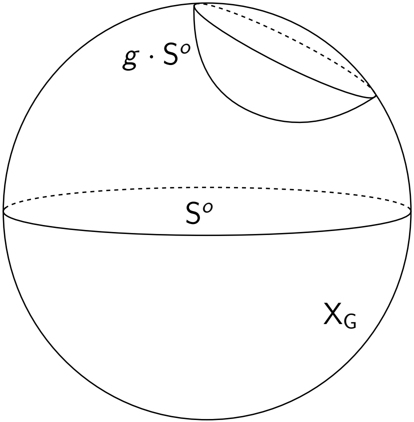

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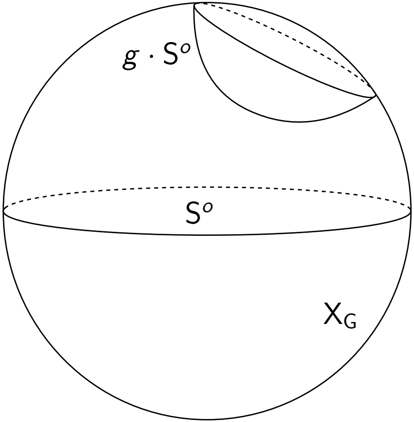

Compute

$$d_{\mathsf{X_G}}(\mathsf{S}^o, g \cdot \mathsf{S}^o) := \inf\{d_{\mathsf{X_G}}(\tau, g \cdot \tau') : \quad \tau, \tau' \in \mathsf{S}^o\}.$$

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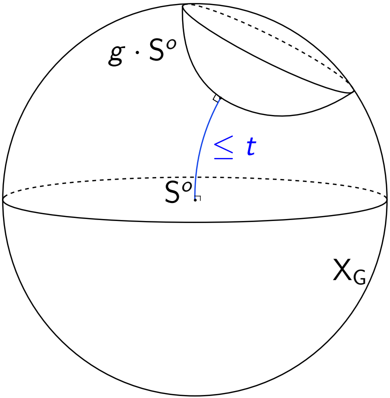

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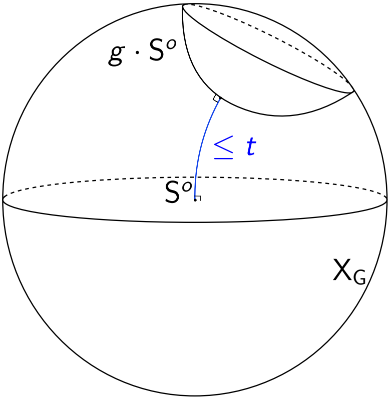

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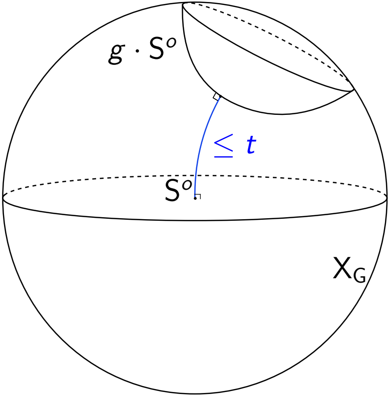

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For $o \in H^{p,q-1}$, understand the asymptotic behaviour as $t \to \infty$ of

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Give some answers for $\Xi < \mathsf{G}$ projective Anosov ($\mathsf{P}_{\{\alpha_1\}}$-Anosov).

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Here: X_{G} contains flats!

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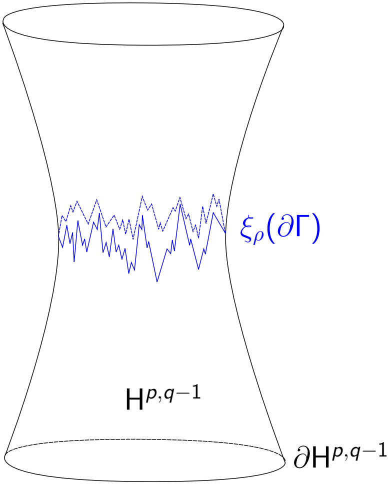
- Uniform "contraction/dilation" property.


$\xi_\rho(\partial\Gamma)$
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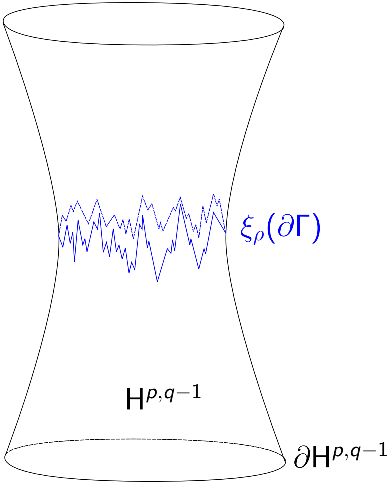
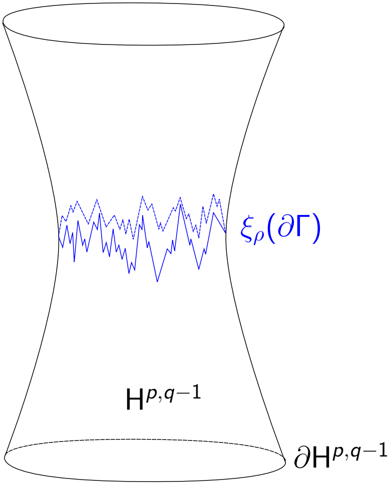
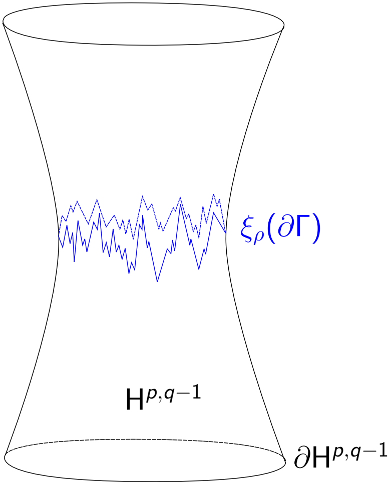


Labourie, Guichard-Wienhard



Labourie, Guichard-Wienhard: (stable class of) quasi-isometric embeddings.



Labourie, Guichard-Wienhard: (stable class of) quasi-isometric embeddings. Higher rank analogue of convex co-compact representations.

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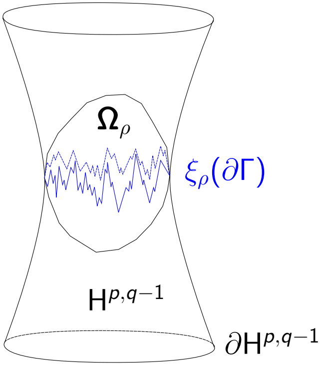

5. Statement of the results

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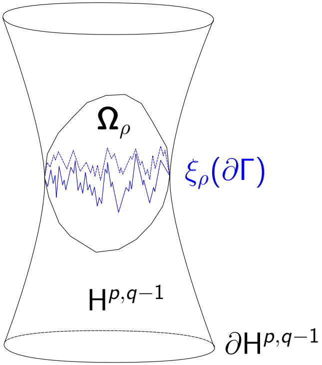

Example

$\mathsf{H}^{p,q-1}$-convex co-compact subgroups of G satisfy $\mathbf{\Omega}_\rho \neq \emptyset$ (c.f. Danciger-Guéritaud-Kassel).

Let $\rho : \Gamma \to \mathsf{G}$ be projective Anosov and

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Recall: $\mathbb{H}^{p,q-1}$ carries a G-invariant metric of signature $(p, q-1)$.

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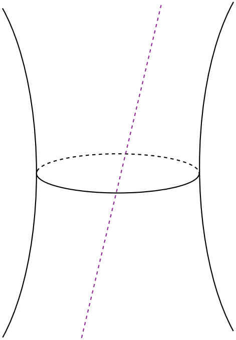
time-like: speed < 0



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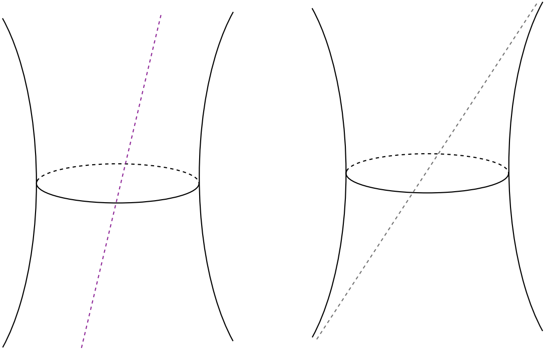

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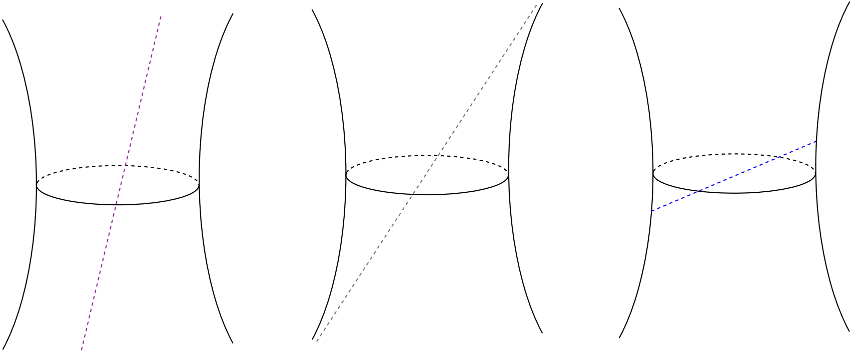

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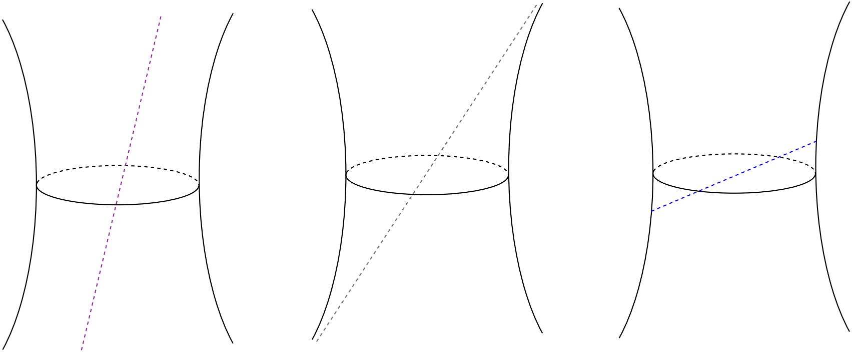

For space-like: can define a length.

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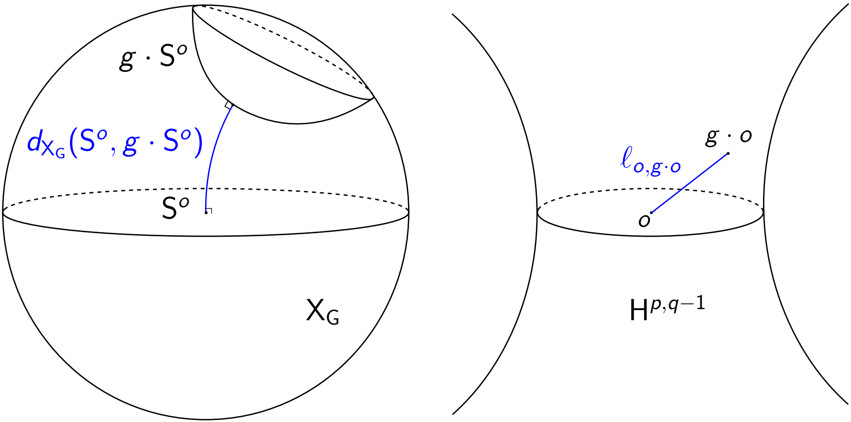

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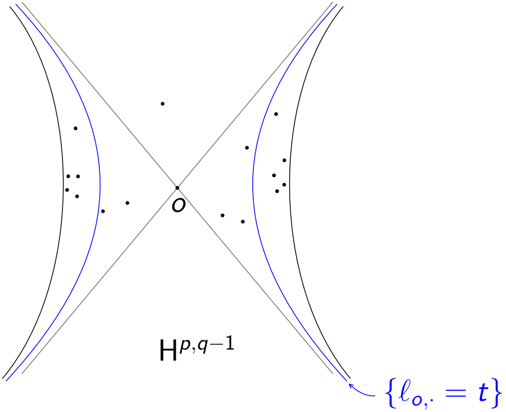

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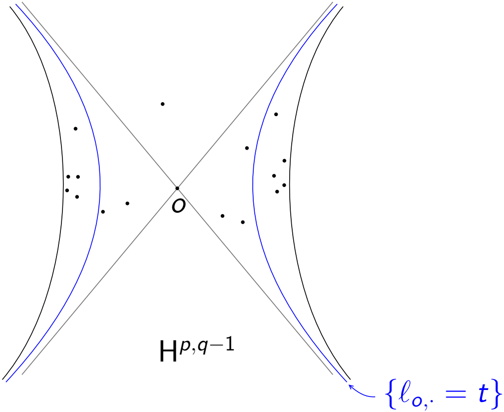

Glorieux-Monclair

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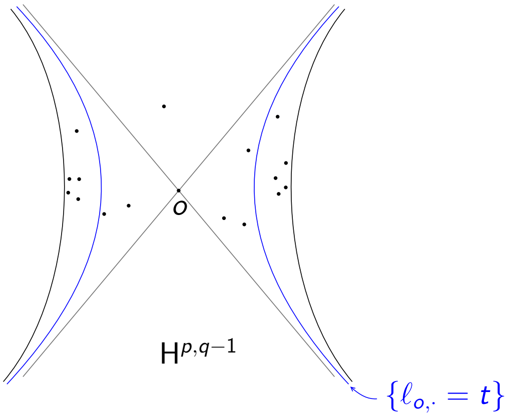

Glorieux-Monclair: study the exponential growth rate of this counting function ("pseudo-Riemannian Hausdorff dimension of the limit set").

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- Compute the $\mathfrak{b}^+$-coordinate using the formula above.

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Structure of the proof:

- Benoist: precise estimate of eigenvalues of products of proximal matrices

$$\tfrac{1}{2}\lambda_1(\sigma^o(\rho\gamma)\rho\gamma^{-1}) \sim \lambda_1(\rho\gamma) + \text{"cross-ratio"}.$$

- Bridgeman-Canary-Labourie-Sambarino: geodesic flow of ρ

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  - $\mathsf{U}\Gamma$ compact quotient of an $\mathbb{R}$-principal bundle over $\partial^2\Gamma$, under some proper action of Γ.
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on $C_c^*(\partial^2\Gamma)$, as $t \to \infty$.

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Purely dynamical result. Challenge: try to use this result + Benoist's estimate to obtain our geometric theorem (use Patterson-Sullivan theory).

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Geometric step: find an explicit μ for which $[\cdot,\cdot]=$"cross-ratio".

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More precisely:

- Find an explicit μ such that

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- Evaluate on $1\in C(\partial\Gamma\times\partial\Gamma)$ ($c:=\tilde{c}^{-1}\|\mu\otimes\mu\|$).

Gracias! Thanks!