Sums of Squares: from real to Commutative Algebra

$p(x)$ nonnegative $\quad p(x) \geq 0$
for all $x \in \mathbb{R}^n$

$\sum q_i^2(x)$

n - # of variables

$2d$ - degree

Thm: Nonnegative polynomials are always sums of squares only in the following cases:

(1) $n = 1$ at most 2 squares

(2) $2d = 2$ n squares

(3) $n = 2$, $2d = 4$ 3 squares

$2d = 2$ Diagonalize # squares
= rank

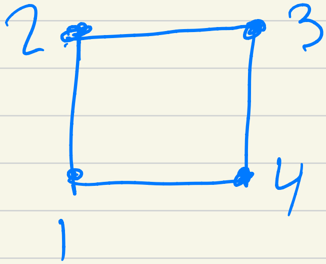
$n = 1$ p nonnegative

$$p = f\bar{f} = (s+it)(s-it)$$

$$f = (s+it) \qquad = s^2 + t^2$$

Example: $I = \langle x_1 x_3, x_2 x_4 \rangle$

$X = V(I)$.



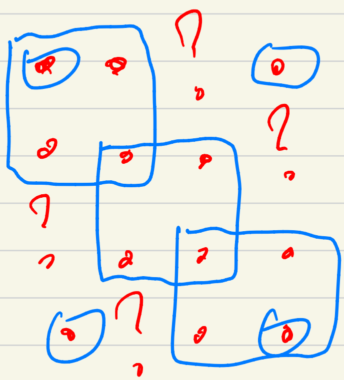
$X = \cup$ Span$\{e_1, e_2\}$ Span$\{e_2 e_3\}$

Span$\{e_3 e_4\}$

Span$\{e_1 e_4\}$

4 lines in $\mathbb{P}^3$

P_X



Σ_X = partially specified matrices that can be completed to a psd matrix

$P_x \supseteq \Sigma_x$ $\quad$ $P_x = \Sigma_x$? $\quad$ $P_x \supsetneq \Sigma_x$

$$\begin{bmatrix} 1 & 1 & ? & -1 \\ 1 & 1 & 1 & ? \\ ? & 1 & 1 & 1 \\ -1 & ? & 1 & 1 \end{bmatrix}$$

squares

$\pi(x)$ = least k such that any form in Σ_x can be written as a sum of k squares

$\pi(x) \leq 3$ $\qquad$ $\pi(x) = 3$

$I = \langle xyz - w^3 \rangle \quad X = \mathcal{V}(I)$

cubic surface in $\mathbb{P}^3$ $\quad \pi(X) = 4$

$\mathcal{P}_X, \ \Sigma_X$ - only globally nonnegative quadratic forms

$$\mathcal{P}_X \underset{\neq}{\supseteq} \Sigma_X$$

$$x^2+y^2+z^2-3w^2$$

$$xyz = w^3 \quad x^2y^2z^2 = w^6$$

$$\frac{x^2+y^2+z^2}{3} - (x^2y^2z^2)^{1/3}$$

$$x = a^3 \quad y = b^2c \quad z = bc^2$$

$$w = abc$$

$$a^6 + b^2c^4 + b^4c^2 - 3a^2b^2c^2 =$$

$$M(a, b, c)$$

$n = 3$ every ternary form is a sum of squares of rational functions

1920's Solved by Artin.

$$p = \sum \left(\frac{q_i}{r_i}\right)^2$$

Enough to consider quadratics:

$$P_{X,2d} = P_{\nu_d(X),2}$$

$$\Sigma_{X,2d} = \Sigma_{\nu_d(X),2}$$

$$\nu_d : \mathbb{P}^n \to \mathbb{P}^{\binom{n+d}{d}-1}$$

$$[x_0 : x_1 : \ldots : x_n] \hookrightarrow \text{to all}$$

monomials of degree d in $n+1$ variables

$$P_{\nu_2(\mathbb{P}^2)} = \Sigma_{\nu_2(\mathbb{P}^2)}$$

When is $\mathbb{P}_X = \Sigma_X$? X-irreducible

Fact: $X \subseteq \mathbb{P}^n$ $v \in X$

X_v - projection of X away from v.

If $\mathbb{P}_X = \Sigma_X$ then $\mathbb{P}_{X_v} = \Sigma_{X_v}$

If project from codim $X - 1$ many points, then we get a hypersurface Y

$$\mathbb{P}_Y = \Sigma_Y$$

$\Rightarrow \deg Y = 2$

$\deg X = \operatorname{codim} X + 1$

Varieties of minimal degree

$\deg Z \geq \operatorname{codim} Z + 1$

X is a VMD $\Longleftrightarrow$ X is 2-regular.

$x_0^2 + \ldots + x_n^2$

Thm: $P_X = \Sigma_X \iff X$ is 2-regular

X - totally real $\subseteq \mathbb{P}^n$.
reduced

Connection 1: P_X, Σ_X - convex cones in R_2
closed

$$P_X^* = \{ \ell \in R_2^* : \ell(p) \geq 0 \text{ for all } p \in P_X \}$$

$$\Sigma_X^* \quad \text{—//—}$$

$\ell_v \quad v \in X \quad \ell_v(p) = p(v)$

$$P_X^* = \text{Cone}\{ \ell_v : v \in X \}$$

$$\Sigma_X^* = \{ \ell \in R_2^* : \ell(q^2) \geq 0 \text{ for all } q \in R_1 \}$$

$\ell \in \Sigma_X^*$ $\quad Q_\ell : R_1 \to \mathbb{R}$

$$Q_\ell(q) = \ell(q^2) \quad \ell \in \Sigma_X^*$$

$$\Updownarrow$$

$$Q_\ell \text{ is PSD.}$$

<u>Fact</u>: Suppose that $\ell \in \Sigma_X^*$ extreme ray, $\ell \notin P_X^*$ then let $W = \ker Q_\ell$

W is a base-point free linear series on X

$$V(W) \cap X = \phi.$$

and $\langle W \rangle_2$ $\quad s \in \langle W \rangle_2$

$e(s) = 0.$

Example: $X = V_2(\mathbb{P}^2)$

$$\langle q_1, q_2, q_3 \rangle \qquad q_1 \cdot q_2 = q_2 \cdot q_1$$

$$\underset{6}{\downarrow} + \underset{6}{\downarrow} + \underset{6}{\downarrow}$$

$$6 + 6 + 6 - 3 = 15$$

$\shortparallel$

dim of ternary quartics

Eisenbud-Huneke-Ulrich

If X is 2-regular then any bpf linear series on X generates R_2

$\Phi_X \neq \Sigma_X$

How big can a base-point-free linear series on X be so that it doesn't generate R_2?

$N_{2,p}$ - $I(X)$ is generated by quadrics & resolution is linear for $p-1$ steps.

If X has $N_{2,p}$ $\Rightarrow$ any bpf linear series on X

$X \subseteq \mathbb{P}^n$

of dimension $n+1-p$

generates R_2

X - square-free monomial ideal.

X - projected rational normal curve

(B, Justin Chen, Jaewoo Jung)

Real & Complex Behavior can be quite different.

$\Pi(X)$ = least k such that any sum of squares in Σ_X can be written as a sum of at most k squares.

$p = \sum q_i^2 \quad p(v) = 0 \quad v \in X$

$$\Downarrow$$

$q_i(v) = 0$ for all i.

q_i are defined on X_v - projection away from v.

$\Pi(X) \geq \Pi(X_v)$

$Y = X_{v_1, \ldots, v_k}$ & $I(Y)_2 = 0$

then $\pi(Y) = n+1-k$

$gp(X) =$ least k such that there exist $v_1, \dots, v_k$ on X &

$Y = X_{v_1,\dots,v_k}$ and $I(Y)_2 = \phi$.

<u>Fact</u>, If X is irreducible $\Rightarrow$ $gp(X)$ is computed by generic pts.

$\pi(X) \geq n+1-gp(X)$

<u>Open Q</u>: Find a variety X where this bound is not tight.

Can lose at most codim X many quadrics in a step.

Open Q : $gp(V_d(\mathbb{P}^n))$

Known for $\mathbb{P}^2$.

Conjecture : $V_d(\mathbb{P}^n)$ lose quadrics as quickly as possible.

true for $\mathbb{P}^2$.

Iarobino-Kanev known for $\mathbb{P}^2$.

$gp(X) \le \operatorname{codim} X$

$gp(X) = \operatorname{codim}(X) \Longleftrightarrow X$ is VMD

$\pi(X) \ge \dim X + 1$

$\pi(X) = \dim X + 1 \Longleftrightarrow X$ is VMD

$$\dim I(X)_2 \leq \binom{gp(X)+1}{2}$$

$gp(X) \geq$ the length of the linear strand of the resolution

$\parallel$

#steps for which linear syzygies exist $+1$.

Sums of Squares and Quadratic Persistence on Real projective varieties