Intro

- F-sings = sings of char. $p > 0$
  defined via Frobenius

  <u>e.g.</u> <u>F-reg</u>, F-rat'l, <u>F-pure</u>, F-inj ...

  today

- Sings in MMP (= minimal model Program)

  = sings appearing in the process of MMP

  fundamental tool in classifying alg. varieties

  These sings are defined in terms of resol of sings (char. 0)

  <u>e.g.</u> terminal, canonical, <u>log terminal</u>, <u>log canonical</u>, rat'l

  today

@ $\exists$ connection between F-sings and sings in MMP

F-reg $\longleftrightarrow$ lt, F-pure $\longleftrightarrow$ lc

F-sings

$(R, \mathfrak{m})$ Noeth. local domain of char. $p>0$

$q = p^e$

$$R^{1/q} = \{ x \in \overline{\mathrm{Frac}(R)} \mid x^q \in R \} \supset R$$

Assume $R^{1/p}$ is a f.g. R-mod.

(F-finite)

Def

(1) R is F-pure $\iff$ $R \hookrightarrow R^{1/p}$ splits as an R-mod hom

i.e.

$$({}^{\exists}\varphi : R^{1/p} \to R \quad 1 \mapsto 1)$$

(2) R is F-regular

$\iff$ $0 \neq {}^{\forall}c \in R$, ${}^{\exists} q = p^e$,

$R \hookrightarrow R^{1/q} \xrightarrow{\times c^{1/q}} R^{1/q}$ splits as an R-mod. hom

$$(\text{i.e. } {}^{\exists}\psi : R^{1/q} \to R \quad c^{1/q} \mapsto 1)$$

(3) Assume R is normal and $\mathbb{Q}$-Gor'n

$R \subset \omega_R \subset \mathrm{Frac}(R)$

${}^{\exists} r \in \mathbb{N}$, $0 \neq {}^{\exists} f \in R$,

$$\omega_R^{(r)} := (\omega_R^{r})^{**} = \frac{1}{f} R$$

reflexive hull

$\mathfrak{a} < R$ ideal, $\lambda \in \mathbb{R}_{>0}$

$$\tau(\mathfrak{a}^\lambda) := \operatorname{Ann}_{\omega_R} \underbrace{0^{*\mathfrak{a}^\lambda f^{\frac{1}{r}}}_{H^d_{\mathfrak{m}}(R)}}_{\text{submodule of } H^d_{\mathfrak{m}}(R)} \subset R \quad \text{test ideal}$$

$z \in H^d_{\mathfrak{m}}(R)$

$$z \in 0^{*\mathfrak{a}^\lambda f^{\frac{1}{r}}}_{H^d_{\mathfrak{m}}(R)} \iff 0 \neq {}^\exists c \in R,$$

$$c\, \mathfrak{a}^{\lceil \lambda q \rceil} f^{\lceil \frac{q}{r} \rceil} F^e(z) = 0 \text{ in } H^d_{\mathfrak{m}}(R)$$

where $F : H^d_{\mathfrak{m}}(R) \longrightarrow H^d_{\mathfrak{m}}(R)$ Frobenius action.

Basic properties

(0) F-reg $\Rightarrow$ F-pure

(1) R is F-reg $\iff \tau(R^1) = R$

$\iff \tau(\mathfrak{a}^1) \supset \mathfrak{a}$ for ${}^\forall \mathfrak{a}$

(2) If $\lambda \leq \mu$

$\Rightarrow \tau(\mathfrak{a}^\lambda) \supset \tau(\mathfrak{a}^\mu)$

(3) If $\mathfrak{a} < \mathfrak{b}$

$\Rightarrow \tau(\mathfrak{a}^\lambda) < \tau(\mathfrak{b}^\lambda)$

<u>Lem</u> If R is F-pure,

$0 \neq {}^{\exists} J \lhd R$, $J \lhd \tau(J^{1-\varepsilon})$ $(0 < {}^{\forall}\varepsilon \ll 1)$

<u>e.g.</u>

(1) $\left(\mathbb{F}_p[X,Y,Z]/X^2+Y^2+Z^2\right)_{(X,Y,Z)}$ $(p \neq 2)$

is F-reg.

(2) $R = \left(\mathbb{F}_p[X,Y,Z]/X^3+Y^3+Z^3\right)_{(X,Y,Z)}$

R is F-pure $\Leftrightarrow$ $p \equiv 1 \mod 3$

Assume $p \equiv 1 \mod 3$

$m := (x,y,z) \lhd R$

Then $\tau(m^{\lambda}) = m^{1+\lfloor\lambda\rfloor}$

In part. $\tau(m^{1-\varepsilon}) = m$, $\tau(m^1) = m^2 \not\ni m$

<u>Sings in MMP</u>

$R = \left(\mathbb{C}[X_1,\cdots,X_n]/(f_1,\cdots,f_r)\right)_{(X_1,\cdots,X_n)}$

Assume R is normal and $\mathbb{Q}$-Goren.

$\pi : Y \to X$ proper birat'l morphism
$X = \operatorname{Spec} R$ w. Y normal
(e.g. resol of sings)

K_Y, K_X canonical div of Y, X

R is $\mathbb{Q}$-Gor $\rightsquigarrow$ $\pi^* K_X$ is defined
$\mathbb{Q}$-Weil div on Y

R is klt (lc) $\Leftrightarrow$ ${}^\forall \pi$, ${}^\forall E$ prime div on Y
$\operatorname{ord}_E (K_Y - \pi^* K_X) > -1$
(≥ -1)

$\mathfrak{a} \subset R$, $\lambda \in \mathbb{R}_{>0}$

We can define the multiplier ideal
$\mathcal{J}(R, \mathfrak{a}^t) \subset R$ similarly

Then $\mathcal{J}(R, R') = R \Leftrightarrow R$ is lt

N.B.

These def make sense in any char.
(even in mixed char.)

Reduction modulo p

$R = \mathbb{C}[x_1, \cdots, x_n]/(f_1, \cdots, f_r)$

$\mathfrak{m} = (x_1, \cdots, x_n) \subset R$

Assume R is normal and $\mathbb{Q}$-Goren

Coeff. of the $f_i \in \mathbb{Z}$

$R_{\mathbb{Z}} := \mathbb{Z}[x_1, \cdots, x_n]/(f_1, \cdots, f_r)$

$\mathfrak{m}_{\mathbb{Z}} := (x_1, \cdots, x_n) \subset R_{\mathbb{Z}}$

$p \in \mathbb{Z}$ prime

$R_p := \mathbb{F}_p[x_1, \cdots, x_n]/(\bar{f}_1, \cdots, \bar{f}_r)$

$\mathfrak{m}_p := \mathfrak{m}_{\mathbb{Z}} R_p$

Compare sing of R at $\mathfrak{m}$ and

sing of R_p at $\mathfrak{m}_p$

$$\begin{array}{ccc} f : \operatorname{Spec} R_{\mathbb{Z}} & \longrightarrow & \operatorname{Spec} \mathbb{Z} \quad \text{flat} \\ \| & & \| \\ \mathcal{X} & & T \ni (p), \underbrace{\mathbb{Z}}_{\text{generic pt}} \end{array}$$

$X_p := f^{-1}(p) = \operatorname{Spec} R_p$

$X_\eta := f^{-1}(\eta) = \operatorname{Spec} \mathbb{Q}[x_1, \cdots, x_n]/(f_1, \cdots, f_r)$

$\operatorname{Spec} R$ = flat base change of X_η

<u>Known result</u>

(1) (Hara-Watanabe '04)

If R_p is $\mathbb{Q}$-Goren F-reg at m_p

(resp. F-pure) for infinitely many p

$\Longrightarrow$ R is lt (resp. lc) at m

(2) (Mehta-Srinivas '97, Hara '98)

If R is lt at m

$\Longrightarrow$ R_p is $\mathbb{Q}$-Goren F-reg at m_p for almost all p

(3) (Ma-Schwede '18)

Assume $R_\mathbb{Z}$ is normal and $\mathbb{Q}$-Goren.

If R_p is $\mathbb{Q}$-Goren. F-reg at m_p for a single p

$\Longrightarrow$ R is lt at m

Thm 1 (K. Sato – T)

Assume $R_{\mathbb{Z}}$ is normal and $\mathbb{Q}$-Goren.

If R_p is $\mathbb{Q}$-Goren. F-pure at m_p for a single p

$\Longrightarrow$ R is lc at m

Thm 1 can be reduced to Thm 1′

Thm 1′

(A, m_A) complete normal $\mathbb{Q}$-Gor. local ring of mixed char. $(0, p)$

$x \in m_A$ NZD

If $A/(x)$ is F-pure of char. p

$\Longrightarrow$ A is lc

Def (Ma - Schwede)

(A, m_A) as above

A $\mathbb{Q}$-Gor $\rightsquigarrow$ $A \subset \omega_A \subset \mathrm{Frac}(A)$

$\exists r \in \mathbb{N}, \ 0 \neq \exists f \in A$

$\omega_A^{(r)} = \frac{1}{f} A$

$0 \neq g \in \mathfrak{m}, \quad n \in \mathbb{N}$

A^+ : int. closure of A in $\overline{\mathrm{Frac}(A)}$

$\ni f^{\frac{1}{p^\infty}}, \; g^{\frac{1}{n}}$

$$\tau_B(A, g^{\frac{1}{n}}) := \mathrm{Ann}_{\omega_A} \underbrace{0^{B, f^{\frac{1}{p^\infty}} g^{\frac{1}{n}}}_{H^d_{\mathfrak{m}_A}(A)}}_{\text{submodule of } H^d_{\mathfrak{m}_A}(A)} \subset A$$

BCM test ideal

$z \in H^d_{\mathfrak{m}_A}(A)$

$z \in 0^{B, f^{\frac{1}{p^\infty}} g^{\frac{1}{n}}}_{H^d_{\mathfrak{m}_A}(A)}$

$\overset{\text{def}}{\Longleftrightarrow}$ $\exists B$ int. perfectoid big CM A^+-alg

$z \in \mathrm{Ker}\left(H^d_{\mathfrak{m}_A}(A) \xrightarrow[\times f^{\frac{1}{p^\infty}} g^{\frac{1}{n}}]{} H^d_{\mathfrak{m}_A}(B)\right)$

<u>Def</u> (Sato - T)

$0 \neq \mathfrak{a} \subset A$ ideal, $\lambda \in \mathbb{R}_{>0}$

$$\tau_B(A, \mathfrak{a}^\lambda) := \sum_{n \geq 1} \sum_{g \in \mathfrak{a}^{\lceil n\lambda \rceil}} \tau_B(A, g^{\frac{1}{n}})$$

Important properties

(i) $\tau_B(A, \mathfrak{a}^\lambda) \subset \mathcal{J}(A, \mathfrak{a}^\lambda)$

(ii) (Restriction thm)

Assume $A/(x)$ is F-finite of char. $p>0$

$\Rightarrow \tau(A/(x), (\mathfrak{a} A/(x))^{\lambda}) \subset \tau_B(A, \mathfrak{a}^{\lambda}) A/(x)$

Thm 1' follows from (i), (ii) and Lem in Page 4.

Q.

In general, $(A, \mathfrak{m})$ Noeth. local ring

$x \in \mathfrak{m}$ NZD

$A/(x)$ is $\mathbb{Q}$-Gor. $\not\Rightarrow$ A is $\mathbb{Q}$-Gor.

What if $R_{\mathbb{Z}}$ is NOT $\mathbb{Q}$-Gor ?

Thm 2 (Sato-T)

Suppose $R_{\mathbb{Z}}$ is NOT nec. $\mathbb{Q}$-Gor.

(we assume R is $\mathbb{Q}$-Gor)

If R_p is $\mathbb{Q}$-Gor. F-reg at $\mathfrak{m}_p$

for a single p

$\Rightarrow$ R is lt at $\mathfrak{m}$

Rem

In the proof of Thm 1, we use $\tau_B(R_{\mathbb{Z}})$

But, in Thm 2, since $R_{\mathbb{Z}}$ is NOT $\mathbb{Q}$-Gor.

$\tau_B(R_{\mathbb{Z}})$ is NOT defined.

We use $\tau_B(\omega_{R_{\mathbb{Z}}}, (\omega_{R_{\mathbb{Z}}}^{(m)})^{1-\frac{1}{m}})$ instead.

So far, we discussed arithmetic deform of F-sing.

Next, we discuss geom. deform of F-sing.

Deformation Problem in CA

$(R, \mathfrak{m})$ Noeth. local ring

$x \in \mathfrak{m}$ NZD

If $R/(x)$ satisfies (P),

does R satisfy (P)?

Deformation Problem in AG

$f: X \to T$ proper flat
$T = \operatorname{Spec} D$ (D is a Dedekind domain)

$T \ni t$ closed pt
$T \ni \eta$ generic pt

If $X_t := f^{-1}(t)$ satisfies (P)
does $X_\eta := f^{-1}(\eta)$ satisfy (P) ?

Assume (R, m) F-finite normal local domain of char. $p > 0$

$D = k[t]$, k perfect field of char. $p > 0$

Known results

(i) $R/(x)$ is $\mathbb{Q}$-Gor. F-reg (resp. F-pure)

$\Rightarrow$ R is F-reg (resp. F-pure)

(Singh '99)

(ii) Assume R is $\mathbb{Q}$-Gor.

$R/(x)$ is $\mathbb{Q}$-Gor. F-reg (resp. F-pure)

$\Rightarrow$ R is F-reg (resp. F-pure)

(Aberbach-Katzman-MacCrimmon '98,

Polstra-Simpson '20)

$\rightsquigarrow$ If X is $\mathbb{Q}$-Gor.

X_t is $\mathbb{Q}$-Gor. F-reg (resp F-pure)

$\Rightarrow$ X_η is F-reg (resp. F-pure)

<u>Thm 3</u> (Sato-T)

Suppose X is NOT nec. $\mathbb{Q}$-Gor.

but X_η is $\mathbb{Q}$-Goren.

X_t $\mathbb{Q}$-Gor. F-reg $\Rightarrow$ X_η F-reg

<u>Rem</u> In dim 2,

F-reg $\Rightarrow$ F-rat'l $\Rightarrow$ $\mathbb{Q}$-Gor.

Thus, if $\dim X_t = 2$,

X_t F-reg $\Rightarrow$ X_η F-reg

(in this case, $\mathbb{Q}$-Gor of X_η is automatically satisfied)

So, 2-dim F-reg. deform in AG sense!