On the quest for F-rational signature

Joint with Kevin Tucker

Notation (R, m) commutative Noetherian w 1
m is unique maximal ideal
domain of characteristic $p > 0$
F-finite

Def $R^{1/p^e} = \{x \in \overline{Q(R)} \mid x^{p^e} \in R\}$
$R \subseteq R^{1/p^e}$
$x \to (x^{p^e})^{1/p^e}$

Def (Kunz) R is F-finite if for some (all) $e > 0$ R^{1/p^e} is finite R-module

Def F-signature (Huneke-Leuschke, Smith-Van den Bergh)

$$s(R) := \limsup_{e \to \infty} \frac{\text{largest } N,\ R^{1/p^e} \twoheadrightarrow \oplus^N R \to 0}{\text{rank}_R R^{1/p^e}}$$

$:= \dim_{Q(R)} R^{1/p^e} \otimes Q(R)$

Nice properties:

1) limsup = lim (Tucker)
2) $s(R) \geq 0$. $s(R) > 0 \iff R$ is strongly F-regular (Aberbach-Leuschke)
3) $s(R) \leq 1$. $s(R) = 1 \iff R$ is regular
$\Leftarrow$ Kunz
$\Rightarrow$ Huneke-Leuschke
4) $s(R)$ is lower semicontinuous
i.e. $\{p \in \operatorname{Spec} R \mid s(R) > a\}$ is open $\forall a$
4') $s(R_p) \geq s(R)$

Main Theorem there exist an invariant $S_{rr}(R)$ w same properties except 2) $S_{rr}(R) > 0 \iff R$ is F-rational

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I Definitions

Def R is strongly F-regular
If for any $c \neq 0$ $\forall e \gg 0$ $\exists \varphi: R^{1/p^e} \to R \to 0$
$c^{1/p^e} \to 1$

Def Tight closure of I
$x \in I^*$ if $\exists c \neq 0$ s.t. $x \underline{\underline{c}}^{1/p^e} R^{1/p^e} \subseteq I R^{1/p^e}$ $\forall e \gg 0$

thm Strongly F-regular $\Rightarrow I^* = I$ $\forall I$

Converse is a conjecture

Def R is F-rational if for some (all) system of parameters $\underline{x} = x_1 \ldots x_d$
$(\underline{x})^* = (\underline{x})$

Thm R is F-rational if and only if

Hochster-Huneke, Gabber, Velez:
R is Cohen-Macaulay
has dualizing module ω_R
$\forall c \neq 0$ $\forall e \gg 0$ $\exists \varphi: \omega_R^{1/p^e} \to \omega_R$
$\varphi(c^{1/p^e} \omega_R^{1/p^e}) = \omega_R$

F-rational $\approx$ str F-regular in ω_R

Cor If R is Gorenstein, then F-rational $\iff$ strongly F-regular

Goal F-signature for F-rationality.

I Hochster-Yao approach

Def Hilbert-Kunz Multiplicity (Monsky)

$$e_{HK}(I) = \lim_{e \to \infty} \frac{\text{length}(R^{1/p^e}/I R^{1/p^e})}{\text{rank}_R R^{1/p^e}}, \quad \sqrt{I} = m$$

Properties If $I \subseteq J$ then $e_{HK}(I) \geq e_{HK}(J)$
If $e_{HK}(I) = e_{HK}(J) \iff J \subseteq I^*$

Thm (Polstra-Tucker, conj Watanabe-Yoshida)
$s(R) = \inf\{e_{HK}(I) - e_{HK}(J) \mid I \subsetneq J,\ \sqrt{I} = m\}$

Def F-rational signature (Hochster-Yao)
$s_{rat}(R) = \inf\{e_{HK}(\underline{x}) - e_{HK}(I) \mid \underline{x}$ SOP, $\underline{x} \subset I\}$

thm ① $\underline{x}$ can be fixed
② $s_{rat}(R) > 0 \iff R$ is F-rational

We don't know much else
$s_{rat}(R) \leq 1$?

Def dual F-signature (Sannai)

$$s_{dual}(R) = \limsup_{e \to \infty} \frac{\text{largest } N\ \ \omega_R^{1/p^e} \twoheadrightarrow \oplus^N \omega_R \to 0}{\text{rank}_R \omega_R^{1/p^e}}$$

Thm 1) $s_{dual} \geq 0$. $s_{dual}(R) > 0 \iff R$ is F-rational
$s_{rat}(R) \geq s_{dual}(R) \geq \frac{s_{rat}(R)}{\text{type } R}$
2) $s_{dual}(R) \leq 1$. $s_{dual}(R) = 1 \iff R$ is regular
3) $s_{dual}(R_p) \geq s_{dual}(R)$

Ex V_n degree n Veronese $k[x, y]$
$s_{dual}(V_n) = 1/2 > s_{rat}(V_n) = 1 - 1/n$
4) R is Gorenstein & F-rational
$\Rightarrow s_{dual}(R) = s(R) > 0$
$\iff s_{rat}(R) - s(R) > 0$

Def Normalized F-rational signature

$$s_{norm}(R) = \inf\left\{\frac{e_{HK}(\underline{x}) - e_{HK}(I)}{\text{col}(\underline{x}) - \text{col}(I)} \;\middle|\; \underline{x} \text{ sop},\ \underline{x} \subsetneq I\right\}$$

Hochster-Yao: $s_{rat}(R_p)\, e\left(\underline{y} R/p\right) \geq s_{rat}(R)$, $\underline{y}$ sop

Properties 1) $s_{rat}(R) \geq s_{norm}(R) \geq s_{dual}(R) \geq s(R)$
Therefore $s_{norm}(R) > 0 \iff R$ is F-rational

and R is Gor F-rat $\Leftrightarrow$ $s_{norm}(R) = s(R) > 0$

2) $s_{norm} \le 1$. $s_{norm}(R) = 1 \Leftrightarrow R$ is regular

3) $s_{norm}(R_p) \ge s_{norm}(R)$

4) $\underline{x}$ can be fixed. Also restricts to $mI \subset (\underline{x})$

Open Problem: $s_{norm}(R) = s_{dual}(R)$?

<u>Pf</u> 2) <u>Kunz</u> R is regular $\Rightarrow R^{1/p^e}$ is R-free

$\Rightarrow e_{HK}(I) = col(I)$

$\Rightarrow s_{norm}(R) = 1$

ETS $s_{norm}(R) \ge 1 \Rightarrow R$ is regular

Take $I = m$

$$\frac{e_{HK}(\underline{x}) - e_{HK}(m)}{col(\underline{x}) - col(m)} \ge s_{norm}(R) \ge 1$$

$$\underbrace{[e_{HK}(\underline{x}) - col(\underline{x})]}_{\overset{\text{Lemma: } \wedge}{0}} + \underbrace{col(m)}_{1} \ge e_{HK}(m)$$

$\Downarrow$

$1 \ge e_{HK}(m) \ge 1$

$1 = e_{HK}(m)$

Watanabe-Yoshida $\Rightarrow$ R is regular

<u>Lemma</u>:

$\dim R = 1$ x is a parameter

$\Rightarrow col(x^n) \le n\, col(x)$

<u>Pf</u> $R/(x^{n-1}) \xrightarrow{1 \mapsto x} R/(x^n) \to R/(x) \to 0$

<u>Cor</u> $col(x_1^{p^e}, \dots, x_d^{p^e}) \le p^{ed} col(x_1, \dots, x_d)$

$\Rightarrow e_{HK}(\underline{x}) \le col(\underline{x})$

<u>Semicontinuity</u>:

<u>Nagata's criterion</u>

$\{p \in \operatorname{Spec} R \mid f(p) > a\}$ is open

$\Updownarrow$

$f: \operatorname{Spec} R \to \mathbb{R}$ is lower semicontinuous

iff 1) $f(p) \ge f(q)$ if $p \subseteq q$

2) If $f(p) > a$ $\exists s \notin p$ st $\forall q \ni p$ $s \notin q$

$f(q) > a$

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1) <u>How</u> to show that $s_{norm}(R_p) \ge s_{norm}(R)$

Take $I \lhd R_p$ and compare w/ $J = (I \cap R, \underline{y})$

st $\underline{y}$ R/p - is sop

$$\frac{e_{HK}(\underline{x}) - e_{HK}(I)}{col(\underline{x}) - col(I)} \underset{\circledast}{\ge} \frac{e_{HK}(\underline{x}, \underline{y}) - e_{HK}(J)}{col(\underline{x}, \underline{y}) - col(J)}$$

Problems with 2: ⓐ "=" in ⊛ is hard due e_{HK}

ⓑ there are more <u>ideals</u>

ⓐ is solved by uniform convergence

<u>Thm</u> $s_{norm}(R) = \lim\limits_{e \to \infty} \dfrac{\inf \left\{ \dfrac{\operatorname{length}\left(\frac{IR^{1/p^e}}{\underline{x}R^{1/p^e}}\right)}{col(\underline{x}) - col(I)} \;\middle|\; \underline{x} \subsetneq I \right\}}{\operatorname{rank}_R R^{1/p^e}}$

<u>& conv is uniform on Spec R</u>

ETS $\underline{\inf} \left\{ \dfrac{\operatorname{length}\left(\frac{IR^{1/p^e}}{\underline{x}R^{1/p^e}}\right)}{\operatorname{length}(I/\underline{x})} \;\middle|\; \begin{array}{l} \underline{x} \subsetneq I \\ mI \le \underline{x} \end{array} \right\}$

is lower semicontinuous

Different Interpretation

<u>Thm</u> <u>Def</u> (Aberbach-Enescu, Tucker)

$$s(R) = \lim_{e \to \infty} \frac{\operatorname{length}\left(R^{1/p^e} / (x^{1/p^e} \mid \forall \varphi: R^{1/p^e} \to R,\ \varphi(x^{1/p^e}) \in m)\right)}{\operatorname{rank}_R R^{1/p^e}}$$

Similarly, Hochster-Yao in (R, m, k)

Correspondence: $\frac{I}{(x)}$ st $mI \subset (x) \Longleftrightarrow V$, st $\omega_R \otimes k \xrightarrow{\pi_V} V \to 0$

Then: $\dim_k \left(\dfrac{IR^{1/p^e}}{xR^{1/p^e}}\right) = \dim_k\ \omega_R^{1/p^e} / (x^{1/p^e} \mid \forall \varphi: \omega_R^{1/p^e} \to \omega_R,\ \pi_V(\varphi(x^{1/p^e})) = 0)$

And we can restate.

$$\inf \left\{ \frac{\dim_k \left(\frac{IR^{1/p^e}}{xR^{1/p^e}}\right)}{\dim_k (I/\underline{x})} \;\middle|\; \begin{array}{l} x \subsetneq I \\ mI \le \underline{x} \end{array} \right\} \underset{(*)}{=} \inf \left\{ \frac{\dim_k \overset{\downarrow}{\text{—}}}{\dim_k V} \;\middle|\; \omega_R \otimes k \to V \to 0 \right\}$$

We can parametrize such V using

the Grassmannian of ω_R on Spec R

This is $G \to \operatorname{Spec} R$, $x \in G \approx \underset{p \in \operatorname{Spec} R}{} \left[\begin{array}{c} k(x) \\ \uparrow \\ k(p) \end{array} \middle| \omega_R \otimes_R k(x) \overset{\pi(x)}{\twoheadrightarrow} V(x) \right]$

Motivated by (*) consider

let $r_1 - r_{n_e}$ such that r_i^{1/p^e} generate R^{1/p^e} as R-module

$$C_e(k(x), V(x)) = \operatorname{rank}\left[\omega_R^{1/p^e} \otimes k(x) \xrightarrow{\oplus r_i^{1/p^e}} \bigoplus^{n_i} \omega_R^{1/p^e} \otimes k(x) \xrightarrow{\oplus \operatorname{Trace}^e} \bigoplus^{n_e} \omega_R \otimes k(x) \to \bigoplus^{n_e} V(x)\right]$$

where $\operatorname{Hom}_R(\omega_R^{1/p^e}, \omega_R) = R^{1/p^e} \operatorname{Trace}^e$

<u>Theorem</u> $x \mapsto \dfrac{C_e(k(x), V(x))}{\dim V(x)}$ is semicontinuous on G

$p \to \inf \left\{ \dfrac{C_e(x)}{\dim V(x)} \;\middle|\; \begin{array}{c} x \\ \downarrow \\ p \end{array} \right\}$ is semicontinuous

taking limit we get new invariant

<u>$s_{Tr}(R)$</u> satisfies all desired propertis

$$s_{rat} \ge \underbrace{s_{norm} \ge s_{Tr}}_{\text{Problem: are equal ??}} \ge s_{dual}$$