# Simplicial ~~Free~~ Resolutions of Powers of (square-free) monomial ideals

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(WICA CCA group, Banff 2019)

$I =$ homogeneous ideal $S = k[x_1, \dots, x_n]$.

$\rightsquigarrow$ (graded) free res.

$$0 \to S^{\beta_d} \to \dots \to S^{\beta_1} \to S^{\beta_0}$$

when minimal, $\beta_i = \beta_i(I)$ ith Betti nr of $I$
$d = pd(I)$

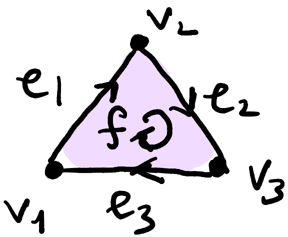
The Taylor resolution of a monomial ideal.

Assume $I = (m_1, \dots, m_q) \leftarrow$ monomials.

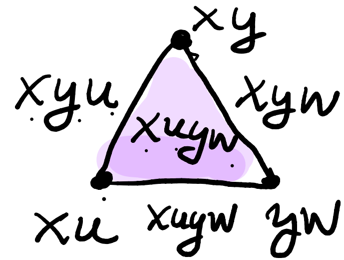
Taylor complex = the labelled simplex on $q$ vertices, labelled with $m_1, \dots, m_q$

induced lcm labelling

$\underline{Ex}$: $I = (xu, xy, yw) \subseteq K[x,y,u,w] \doteq S$




label $\rightsquigarrow$




Simplicial chain complex $\rightsquigarrow$ Free multigraded res. of $I$

homogenization

$$0 \to K \to K^3 \to K^3$$

$$f \qquad e_1, e_2, e_3 \qquad v_1, v_2, v_3$$

$$\partial(f) = e_1 + e_2 + e_3$$

$$\partial(e_1) = v_2 - v_1$$

$$\partial(e_2) = v_3 - v_2$$

$$\partial(e_3) = v_1 - v_3$$

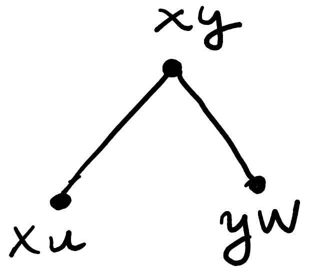
$$0 \to S(xuyw) \to \begin{matrix} S(xyu) \\ \oplus \\ S(xyw) \\ \oplus \\ S(xuyw) \end{matrix} \to \begin{matrix} S(xu) \\ \oplus \\ S(xy) \\ \oplus \\ S(yw) \end{matrix}$$

$$\underline{\underline{\partial(f)}} = \frac{\cancel{xuy}w}{\cancel{xyu}} e_1 + u\, e_2 + 1\, e_3$$

$\hookrightarrow$ not minimal

$$0 \to S \to S^3 \to S^3.$$

Remove the face and the edge with the lcm:




$\rightsquigarrow \quad 0 \to S^2 \to S^3$

minimal free res. of $I$

Lyubeznik resolution of $I = (u_1, u_2, \dots, u_q)$

- order the generators $u_1 < u_2 < \dots < u_q$
- From the Taylor complex build a subcomplex

$\mathbb{L}$ : $\{u_{a_1}, \dots, u_{a_k}\} \in \mathbb{L} \iff u_j \nmid \mathrm{lcm}(u_{a_1}, \dots, u_{a_k})$ $\forall j < a_1$.

$a_1 < a_2 < \dots < a_k$

$\Downarrow$

$\mathrm{lcm}(u_{a_1}, \dots, u_{a_k}) \neq \mathrm{lcm}(\dots, u_j)$

**Thm** (Lyubeznik): $\mathbb{L}$ supports a free res. of $I$.

Our problem: $I = (m_1, \dots, m_q)$
↳ square-free.

Look at $I^2, I^3, \dots$ → The Taylor resolution becomes highly non-minimal.

<u>Goal</u>: <u>Trim</u> the Taylor complex to a smaller simplicial complex that will give better bounds for $\beta_i(I^r)$ and $pd(I^r)$.

<u>CRITERION</u>. (Bayer-Peeva-Sturmfels)
If $\Delta$ is a simplicial complex labelled with monomials $m_1, \dots, m_q$, then the homogenization $F_\Delta$ of chain complex of $\Delta$ supports a res. if and only if

$$\Delta_m = \{ \sigma \in \Delta : \underset{\substack{\text{monomial label} \\ \text{of } \sigma}}{m_\sigma} \mid m \}$$

is acyclic over $k$ $\forall\, m \in LCM(I)$

$q=3$

$r=2 \leftarrow I^2$

$I = (m_1, m_2, m_3)$.

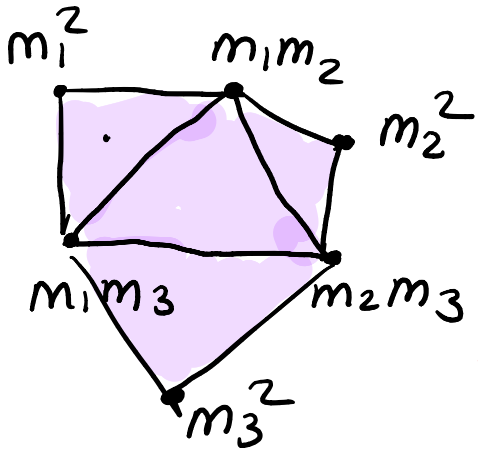
$I^2 = (m_1^2, m_2^2, m_3^2, m_1m_2, m_1m_3, m_2m_3)$

Observe: $m_i m_j \mid \operatorname{lcm}(m_i^2, m_j^2)$.

$m_i m_k \mid \operatorname{lcm}(m_i^2, m_j m_k)$

$\forall i, j, k$ are distinct.

Starting with the Taylor complex on 6 vertices we remove the edges $\{m_1^2, m_2^2\}$, $\{m_1^2, m_3^2\}$, $\{m_2^2, m_3^2\}$, $\{m_1^2, m_2m_3\}$, $\{m_2^2, m_1m_3\}$, $\{m_3^2, m_1m_2\}$.




$\rightsquigarrow$ We can see this supports a res of $I^2$.

$\Longrightarrow \operatorname{pd}(I^2) \leq 2$

If $m_1 = xab$
$m_2 = yac$
$m_3 = zbc$
then $I^2$ has Betti sequence
$6, 9, 4$.

effective

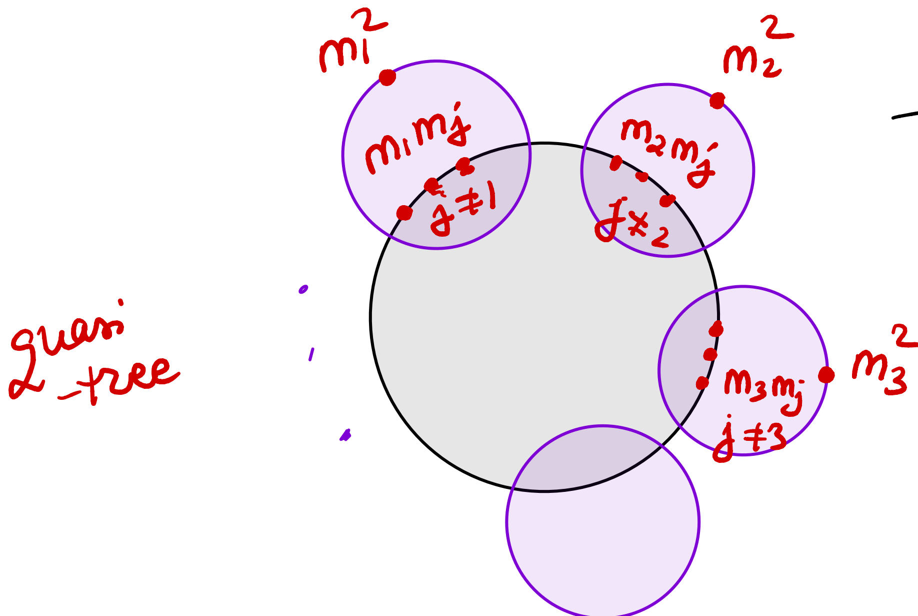
Bounds on the Betti numbers:

$\beta_0 \leq 6$, $\beta_1 \leq 9$, $\beta_2 \leq 4$, $\beta_3 = 0$

For comparison, the simplex on 6 vertices gives:

$\beta_0 \leq 6$, $\beta_1 \leq 15$, $\beta_2 \leq 20$, $\beta_3 \leq 15$, $\beta_4 \leq 6$, $\beta_5 \leq 1$

For any $q$, we define the simplicial complex $\mathbb{L}^2_q$




→ We can show this supports a free resolution of $I^2$

If the generating set of $I^2$ is non-minimal.
{ we can remove the non-minimal ones.

⇝ $\mathbb{L}^2_q(I)$

This still supports a resolution.

<u>Note</u>: There exists an ideal $I$ such that $\beta_i(I^2) = \#(i\text{-dimensional faces of } \mathbb{L}^2_q)$ (res. is minimal)

What makes this work?

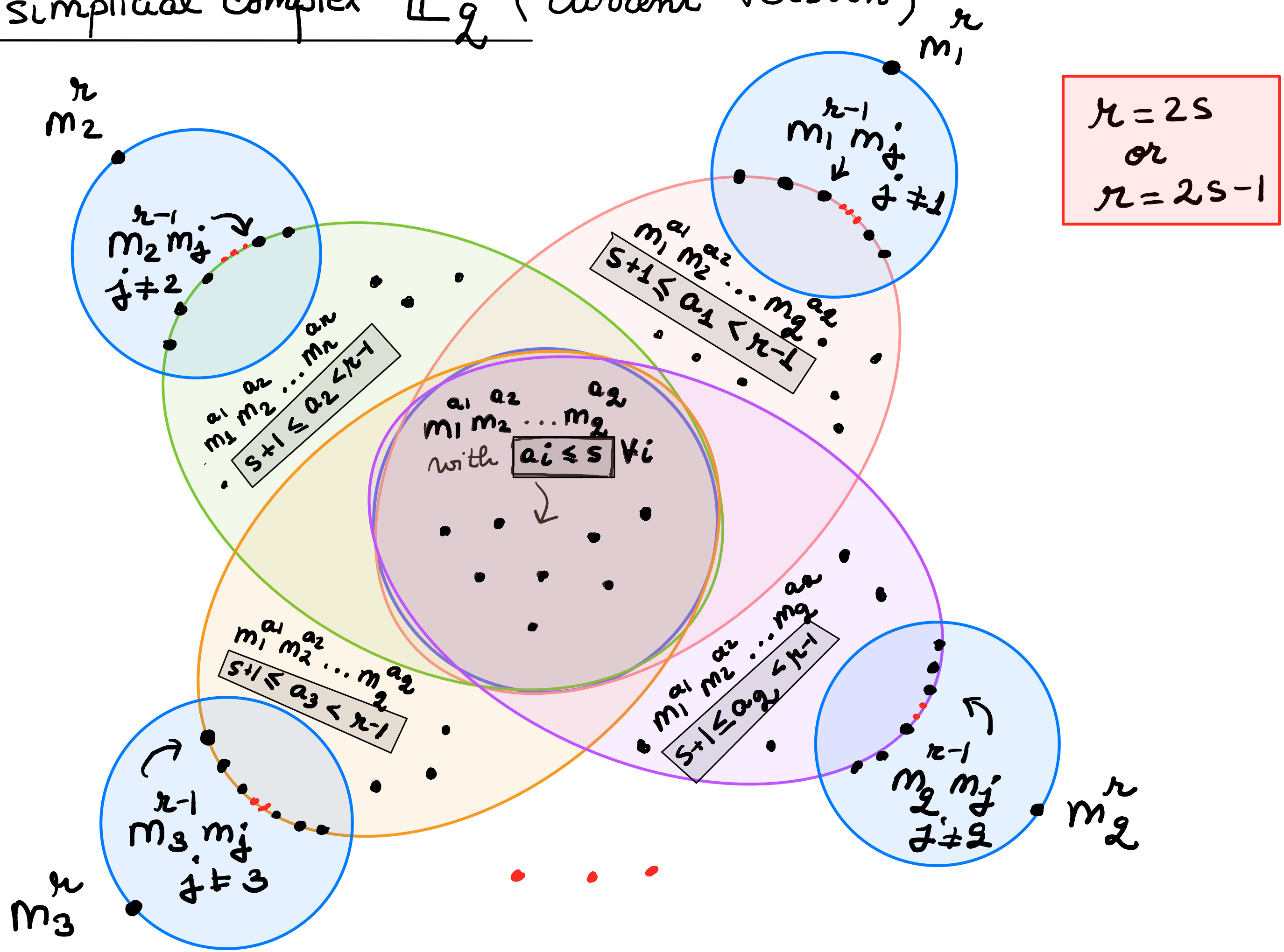
- BPS criterion.
- simplicial collapsing.

- A face $\sigma$ in a simplicial complex $\Delta$ is called a **free face** if it is contained in but not equal to a unique **facet** $F$.

- A **collapse** of $\Delta$ along $\sigma$ is obtained via removing the facets $\tau$ such that $\sigma \subseteq \tau \subseteq F$ from $\Delta$. (Whitehead)

$\rightsquigarrow$ The resulting simplicial complex is homotopy equivalent to $\Delta$.

# The simplicial complex $\mathbb{L}_q^r$ (current version)



$\rightsquigarrow$ Delete the non-minimal gens., according to a certain rule.

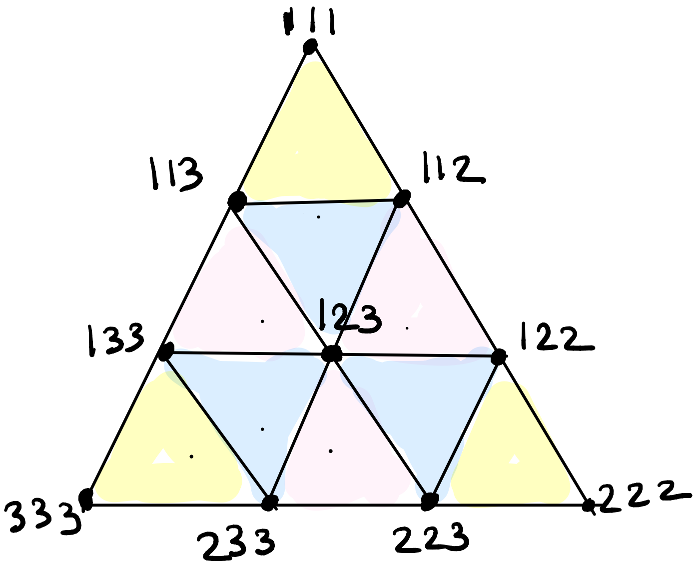
The resulting complex is denoted $\mathbb{L}_2^r(I)$.

$\boxed{\begin{array}{l}\text{THEOREM (WICA CCA group).} \\ \mathbb{L}_2^r(I) \text{ supports a free resolution of } I^r.\end{array}}$

When $r>2$, $\mathbb{L}_2^r$ cannot be minimal.

## CAN WE DO BETTER?

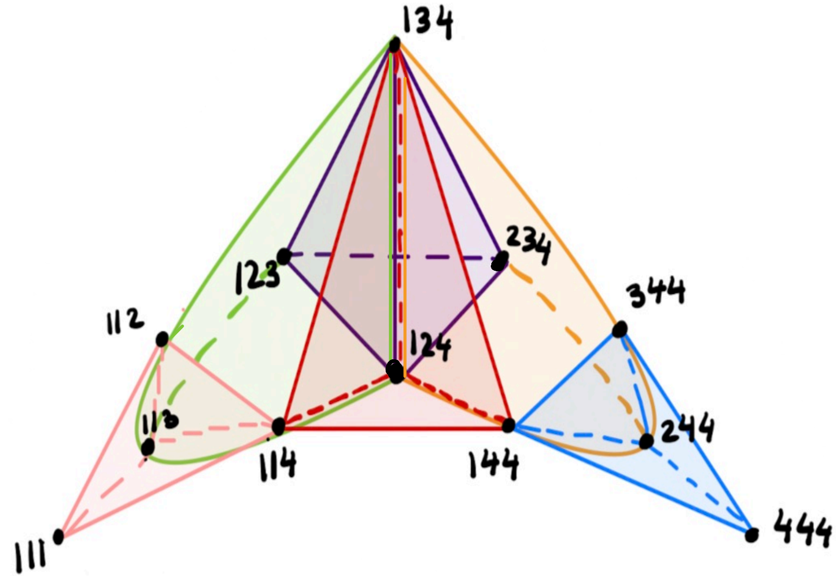
Trimmed $\mathbb{L}_3^3$:




$\Longrightarrow$ $\boxed{pd(I^3) \leq 2 \text{ when } g = 3}$

(Taylor complex gives $\leq 9$)

Incomplete picture of trimmed $\mathbb{L}_4^3$:




$\Longrightarrow$ $\boxed{pd(I^3) \leq 5 \text{ when } g = 4}$

(Taylor complex gives $\leq 19$)