Smooth Hilbert Schemes

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Algebraic Context

DREAM: Describe all ideals in a commutative ring.

FOCUS: Let $R := \mathbb{Z}[x_0, x_1, \dots, x_m]$ be the standard $\mathbb{Z}$-graded polynomial ring. Consider a saturated homogeneous ideal I in R that is flat over $\mathbb{Z}$.

(saturation) $I = I : \langle x_0, x_1, \dots, x_m \rangle^\infty$

$$= \left\{ f \in R \,\middle|\, f\, R_n \subseteq I \text{ for some } n \in \mathbb{N} \right\}$$

(flatness) For all $j \in \mathbb{Z}$, the homogeneous part $(R/I)_j = (R_j / I_j)$ is a free $\mathbb{Z}$-module.

HILBERT POLYNOMIAL: There exists $p \in \mathbb{Q}[t]$ such that $p(j) = \operatorname{rank}(R/I)_j$ for all $j \gg 0$.

Geometric Setting

Bijection: Saturated homogeneous ideals in R correspond to subschemes in $\mathbb{P}^m = \mathrm{Proj}(R)$.

Grothendieck (1961): $\mathrm{Hilb}^p(\mathbb{P}^m)$ parametrizes subschemes in $\mathbb{P}^m$ with Hilbert polynomial p:

- each I is identified with a point in $\mathrm{Hilb}^p(\mathbb{P}^m)$,
- deformations correspond to nearby points,
- one-parameter families correspond to curves.

Challenge: Describe the geometry of the projective scheme $\mathrm{Hilb}^p(\mathbb{P}^m)$.

Combinatorial Beauty

MACAULAY (1926): $\operatorname{Hilb}^p(\mathbb{P}^m) \neq \varnothing$ if and only if there exists $\lambda := (\lambda_1, \lambda_2, \dots, \lambda_r) \in \mathbb{N}^r$ such that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r \geqslant 1$ and $q(t) = \sum_{i=1}^{r} \binom{t+\lambda_i-i}{\lambda_i-1}$.

HARTSHORNE (1966): Each nonempty $\operatorname{Hilb}^p(\mathbb{P}^m)$ is path connected.

GOTZMANN (1978): The Castelnouvo–Mumford regularity of each ideal I parametrized by $\operatorname{Hilb}^p(\mathbb{P}^m)$ is bounded above by r.

REEVES–STILLMAN (1997): Each nonempty $\operatorname{Hilb}^p(\mathbb{P}^m)$ has a smooth point.

Classic Examples

- I is generated by $m-\lambda_1+1$ linear forms if and only if $p(t)=\binom{t+\lambda_1-1}{\lambda_1-1}$, so $r=1$ implies that
$$\mathrm{Hilb}^p(\mathbb{P}^m)=\mathrm{Gr}(\lambda_1-1,\mathbb{P}^m).$$
- $I=\langle f\rangle$ is principal with $\deg(f)=r$ if and only if $p(t)=\sum_{i=1}^{r}\binom{t+m-i}{m-1}$, so $\lambda=(m^r)=(m,m,\ldots,m)$ implies that $\mathrm{Hilb}^p(\mathbb{P}^m)=\mathbb{P}^{\binom{r+m}{m}-1}$.
- When $\lambda=(2^3,1^1)$ or $p(t)=3t+1$, $\mathrm{Hilb}^p(\mathbb{P}^3)$ has two irreducible components: a twisted cubic curve and a plane cubic union a point.

Monstrous Features

MUMFORD (1962): An irreducible component of $\mathrm{Hilb}^{14t-23}(\mathbb{P}^3)$ is generically nonreduced (singular at every point).

ELLIA–HIRSCHOWITZ–MEZZETTI (1992): The number of components of $\mathrm{Hilb}^{dt+1-g}(\mathbb{P}^3)$ is not bounded by a polynomial in d and g.

VAKIL (2006): Every singularity type appears in some $\mathrm{Hilb}^{p}(\mathbb{P}^4)$.

AIM: Classify and understand all smooth Hilbert schemes $\mathrm{Hilb}^{p}(\mathbb{P}^m)$.

Complete Classification

Theorem (Skjelnes–Smith): $\mathrm{Hilb}^p(\mathbb{P}^m)$ is smooth if and only if one of the following holds:

(1) $m = 2 \geqslant \lambda_1$

(2) $m \geqslant \lambda_1$ and $\lambda_r \geqslant 2$,

(3) $\lambda = (1)$ or $\lambda = (m^{r-2}, \lambda_{r-1}, 1)$ where $r \geqslant 2$ and $m \geqslant \lambda_{r-1} \geqslant 1$,

(4) $\lambda = (m^{r-s-3}, \lambda_{r-s-2}^{s+2}, 1)$ where $r-3 \geqslant s \geqslant 0$ and $m-1 \geqslant \lambda_{r-s-2} \geqslant 3$,

(5) $\lambda = (m^{r-s-5}, 2^{s+4}, 1)$ where $r-5 \geqslant s \geqslant 0$,

(6) $\lambda = (m^{r-3}, 1^3)$ where $r \geqslant 3$,

(7) $\lambda = (m+1)$ or $r = 0$.

Ideas of Proof

$\Leftarrow$: Work of Fogarty (1968), Staal (2020), and Ramkumar (2019) shows (1)–(7) are sufficient.

$\Rightarrow$: When $\lambda = (\lambda_1^{r-s-2}, 1^{s+2})$ or λ has at least three distinct parts with $\lambda_{r-1} = \lambda_r = 1$, an explicit monomial ideal is a singular point on $\mathrm{Hilb}^p(\mathbb{P}^m)$.

When $\lambda = (2^2, 1^1)$ or $\lambda = (2^3, 1^1)$, $\mathrm{Hilb}^p(\mathbb{P}^m)$ has two components; the gap between (4) and (5).

When $\lambda = (1^{s+4})$, a known monomial ideal corresponds to a singularity on $\mathrm{Hilb}^{s+4}(\mathbb{P}^m)$.

Residual Flags

Definitions: An inclusion $Y \subset X$ in $\mathbb{P}^n$ is d-residual if there is a hypersurface D in $\mathbb{P}^n$ of degree d such that $I_X = I_Y \cdot I_D$. A residual flag of type $(n_1, d_1), (n_2, d_2), \ldots, (n_e, d_e)$ is a chain $\varnothing \subset X_e \subset X_{e-1} \subset \cdots \subset X_1$ such that X_i lies in some $\mathbb{P}^{n_i}$ and $X_{i+1} \subset X_i$ is d_i-residual in $\mathbb{P}^{n_i}$.

Example: A residual flag of type $(3,2), (2,4)$ in $\mathbb{P}^3$ is a pair $X_2 \subset X_1$ where X_2 is a planar quartic curve and X_2 is a 2-residual scheme in X_1; $I_{X_1} := \langle f_2 \rangle \cap \langle f_1, f_4 \rangle$ where $\deg(f_i) = i$.

New Parameter Space

Proposition: Residual flags of type $(\vec{n}, \vec{d})$ are parametrized by a projective bundle over the partial flag variety of type $\vec{n}$, so its cohomology ring has an explicit presentation.

Partitons: Repackage the type of a residual flag as the partition $\lambda := (n_1^{d_1}, n_2^{d_2}, \dots, n_e^{d_e})$.

Hilbert Polynomial: For any residual flag $\varnothing \subset X_e \subset \cdots \subset X_1$ of type $(\vec{n}, \vec{d})$ in $\mathbb{P}^m$, we have $p_{X_1}(t) = \sum_{i=1}^{r} \binom{t+\lambda_i - i}{\lambda_i - 1}$ where $r := d_1 + \cdots + d_e$.

Geometric Reinterpretation

PROPOSITION: Points on all $\mathrm{Hilb}^p(\mathbb{P}^m)$ satisfying (2) and (3) are residual flags.

COROLLARY: When $\lambda_1 > \lambda_2 > \cdots > \lambda_r > 1$, $\mathrm{Hilb}^p(\mathbb{P}^m)$ is simply a partial flag variety.

PROPOSITION: $\mathrm{Hilb}^p(\mathbb{P}^m)$ satisfying (4)–(6) are birational to the product of the parameter space for residual flags and $\mathbb{P}^m$. A general point on such $\mathrm{Hilb}^p(\mathbb{P}^m)$ corresponds to the disjoint union of a residual flag and a point.