How short can a module of finite projective dimension be?

w/ Srikanth Iyengar & Liquan Ma

$(R, \mathfrak{m}, k)$ local ring, usually Cohen-Macaulay

k infinite.

What is $\min\{\operatorname{length}_R M \mid \operatorname{pd}_R M < \infty\}$?

① R CM $\iff$ $\exists M$ $\operatorname{pd}_R M < \infty$

$\operatorname{length}_R M < \infty$, $M \neq 0$

② $\underline{x} = x_1, \dots, x_d$ s.o.p of R

$\operatorname{pd}_R(R/\underline{x}) < \infty$ &

$\operatorname{length}_R(R/\underline{x}) \geq e(R) =$ Hilbert-Samuel mult. of R

(3) If $\underline{x}$ is sufficiently general then

$$\text{length}_R(R/\underline{x}) = e(R)$$

(4) $Q \hookrightarrow R$ finite flat, Q is regular local, N finite length Q-module, $M = R \otimes_Q N$

Then $\operatorname{pd}_R M < \infty$ &

$$\text{length}_R(M) = \text{length}_Q(N) \cdot \text{length}_R\left(\frac{R}{\mathfrak{m}_Q \cdot R}\right)$$

$$\geq \text{length}_Q(N) \cdot e(R)$$

Bold Conjectures $\cancel{\operatorname{pd}_R M < \infty}$

$\Rightarrow$ (A) $\text{length}_R(M) \geq e(R)$

(B) $\forall i,\ \binom{\dim R}{i} \cdot \text{length}_R(M) \geq \beta_i^R(M) \cdot e(R)$

(C) $\forall i,\ \binom{\dim R}{i}\, \chi_\infty(M) \geq \beta_i^R(M) \cdot e(R)$

Dutta mult. To be defined.

(D) R not nec. CM,

$F_\cdot = (0 \to F_d \to \cdots \to F_0 \to 0)$

$d = \dim R$, $\text{length}_R H_i(F_\cdot) < \infty \quad \forall i$

$$\chi_\infty(F_\cdot) \geq e(R)$$

Ex All 4 hold if $\dim R = 1$

$\text{pd}_R M < \infty$, $\text{length}_R M < \infty$

$\Rightarrow \quad 0 \to R^n \xrightarrow{A} R^n \to M \to 0$

is min'l res'n of M

Fact: $\text{length}_R(M) = \text{length}_R\left(\frac{R}{\det A}\right)$

$\det A \in m^n \Rightarrow$

$\text{length}\left(\frac{R}{\det(A)}\right) \geq n \cdot e(R)$

$\therefore$ A + B hold, $n = \beta_0^R M = \beta_1^R(M)$

C + D both hold

$\chi_\infty(M) = \text{length}_R M$ if $\dim R \leq 2$

An R-module U is Ulrich if

① U is MCM $\left(\Rightarrow e(U) \geq \nu(U) \right.$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \shortparallel$

$\qquad\qquad\qquad\qquad\qquad\qquad \left. \dim_k \left(U/m \cdot U \right) \right)$

② $e(U) = \nu(U)$.

② $\Leftrightarrow$ $\underline{x} \cdot U = m \cdot U$ for a sufficiently general sop $\underline{x}$ of R

$\therefore$ U Ulrich iff

$$H_i\left(\mathrm{Kos}_R(\underline{x}) \otimes_R U \right) = \begin{cases} 0 & i \neq 0 \\ \dfrac{U}{m \cdot U} & i = 0 \end{cases}$$

for $\underline{x}$ suff. general sop.

iff $\mathrm{Kos}_R(\underline{x}) \otimes_R U \sim \dfrac{U}{m \cdot U} \cong k^{\nu(U)}$

R is Ulrich iff R is regular

<u>Theorem</u> [Brennan–Herzog–Ulrich]
If R is a strict complete intersection (e.g., a hypersurface), then R admits a non-zero Ulrich module U s.t. $[U] \in \mathbb{Z}\cdot[R]$ in $G_0(R)$.

<u>Ex</u> $R = k[x,y,z,w]/(xw-yz)$

$A = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ $B = \begin{bmatrix} w & -y \\ -z & x \end{bmatrix}$

$$\begin{tikzcd}
\cdots \arrow[r, "B"] & R^2 \arrow[rr, "A"] \arrow[dr] & & R^2 \arrow[rr, "B"] \arrow[dr] & & R^2 \arrow[r, "A"] & R^2 \arrow[r, "B"] & \cdots \\
& 0 \arrow[r] & N \arrow[ur] \arrow[dr] & 0 \arrow[r] & M \arrow[ur] \arrow[dr] & & & \\
& & & 0 & & 0 & &
\end{tikzcd}$$
is exact

- M, N MCM
- $\mu(M) = 2 = \mu(N)$
- $0 \to N \to R^2 \to M \to 0$ $\Rightarrow$

$$e(N) + e(M) = e(R^2) = 4 \quad (e(R)=2)$$

$$e(N) \geq \nu(N) \qquad e(M) \geq \nu(M)$$

$$\therefore \quad e(M) = \nu(M) \qquad e(N) = \nu(N)$$

$\therefore$ M, N and $M \oplus N$ are Ulrich

<u>Note</u>: $[M \oplus N] = 2 \cdot [R] \in G_0(R)$

U Ulrich $\iff$ $\operatorname{Kos}_R(\underline{x}) \otimes_R U \simeq \dfrac{U}{m \cdot U}$

for a gen'l $\underline{x}$

A sequence of R-modules $U_1, U_2, U_3, \ldots$ is a <u>lim Ulrich sequence</u> if

$$\lim_{n \to \infty} \frac{\operatorname{length} H_i(\operatorname{Kos}(\underline{x}) \otimes_R U_n)}{\nu(U_n)} = 0 \text{ for } i \neq 0$$

and

$$\lim_{n \to \infty} \frac{\operatorname{length}_R \left(\ker\left(\frac{U_n}{\underline{x} U_n} \to \frac{U_n}{m \cdot U_n}\right)\right)}{\nu(U_n)} = 0$$

$\left(\Leftrightarrow \lim_{n\to\infty} \frac{e(U_n)}{\sigma(U_n)} = 1 \right)$

<u>Theorem</u> [IMW] If R has a non-zero Ulrich module U s.t.

$$[U] \in \mathbb{Z}\cdot[R] \in G_0(R), \text{ then}$$

$$\forall i: \quad \binom{d}{i} \operatorname{length}_R(M) \geq \beta_i^R(M)\cdot e(R)$$

for all finite length M s.t. $\operatorname{pd}_R M < \infty$.

More generally, this holds if R has a lim Ulrich sequence $\{U_n\}$ s.t.

$$0 \neq \lim_{n\to\infty} \frac{[U_n]}{\sigma(U_n)} \in \mathbb{Q}\cdot[R] \in G_0(R)_{\mathbb{Q}}$$

<u>Proof</u> (when $\exists\, U$)

$K_\cdot = \operatorname{Kos}_R(\underline{x})$ $\underline{x}$ a gen'l s.o.p.

$F_\cdot$ = min'l free res'n of M

$(\beta_i^R(M) = \operatorname{rank}_R(F_i))$

since U is Ulrich

U MCM

$\operatorname{pd} M < \infty$

$$\mathcal{U}/m\mathcal{U} \otimes_R F_\cdot \simeq K_\cdot \otimes_R \mathcal{U} \otimes_R F_\cdot \simeq K_\cdot \otimes_R \underline{(\mathcal{U} \otimes_R M)}$$

$$\cong$$

$$(F_\cdot \otimes_R h)^{\nu(\mathcal{U})}$$

$$\nu(\mathcal{U}) \cdot \beta_i^R(M) = \text{length } H_i(K_\cdot \otimes_R \mathcal{U} \otimes_R F_\cdot) \le \binom{d}{i} \cdot \text{length}(\mathcal{U} \otimes M)$$

Now, length $(\mathcal{U} \otimes_R M)$ depends only on

$[\mathcal{U}] \in G_0(R)$. $[\mathcal{U}] \in \mathbb{Q} \cdot [R]$

$$\therefore \text{length}_R(\mathcal{U} \otimes_R M) = \frac{e(\mathcal{U}) \cdot \text{length}_R(M)}{e(R)}$$

$$\therefore \cancel{\nu(\mathcal{U})} \cdot \beta_i^R(M) \le \binom{d}{i} \frac{\cancel{e(\mathcal{U})} \cdot \text{length}(M)}{e(R)}$$

$$\nu(\mathcal{U}) = e(\mathcal{U})$$

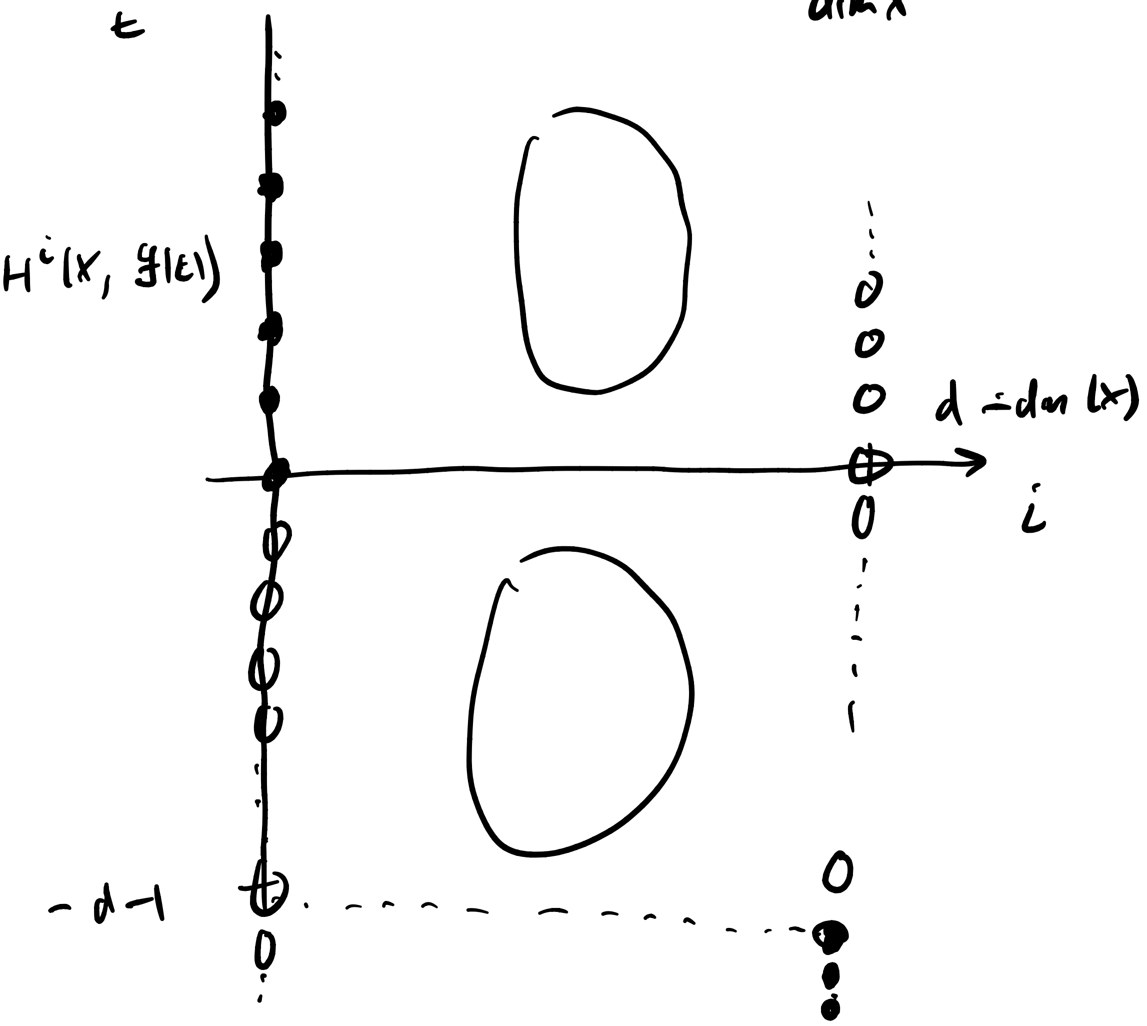
$\boxtimes$

<u>Corollary</u> Conj. A + B hold for strict complete intersections.

An Ulrich sheaf on a projective k-scheme $X \subseteq \mathbb{P}^N_k$ is a coherent sheaf $\mathcal{F}$ on X s.t.

$$H^i(X, \mathcal{F}(t)) \neq 0 \iff \begin{array}{c} (i=0 \text{ and } t \geq 0) \\ \text{or} \\ (i = d \text{ and } t \leq -d-1) \end{array}$$

$d = \dim X$



$\vdots$

Ex $X = \mathbb{P}^d_k$ $\quad \mathcal{F}$ is Ulrich $\Leftrightarrow$ $\mathcal{F} \cong \mathcal{O}_{\mathbb{P}^d_k}^r$

Fact $X = \operatorname{Proj} R$, R = standard graded k-algebra, $\mathcal{F}$ Ulrich sheaf on X $\Rightarrow$ $\Gamma_*(\mathcal{F}) = \bigoplus_t H^0(X, \mathcal{F}(t))$ is a (graded) Ulrich R-module.

Prop [IMW] If $\mathcal{F}_1, \mathcal{F}_2, \ldots$ is a "lim Ulrich sequence" of sheaves on X then $\Gamma_*(\mathcal{F}_1), \Gamma_*(\mathcal{F}_2), \ldots$ is a lim Ulrich sequence of graded R-modules.

Approximate Def:

- 3 minor technical conditions

- $$\lim_{n\to\infty} \frac{\dim_k H^i(X, \mathcal{F}_n(t))}{\dim_k H^0(X, \mathcal{F}_n)} = 0$$ υ

unless $(i=0$ and $t\geq 0)$ or

$\big(i = \underbrace{\dim X}_{= d}$ and $t \leq -d-1\big)$

<u>Theorem</u> [Ma, IMW] k perfect field of char $p>0$, R standard graded k-alg., $X = \operatorname{Proj}(R)$. Then X has a lim Ulrich seq. of sheaves

$\mathcal{F}_1, \mathcal{F}_2, \ldots$ and thus

$\Gamma_*(\mathcal{F}_1), \Gamma_*(\mathcal{F}_2), \ldots$ is a lim Ulrich sequence of graded R-modules. Moreover,

$$\lim_{n\to\infty} [\Gamma_*(\mathcal{F}_n)] \cdot \frac{e(R)}{\upsilon(\Gamma_*(\mathcal{F}_n))} = [R]_{\dim(R)}$$

$$\in G_0(R)_{\mathbb{Q}}$$

$\boxed{d = \dim X}$

to be defined

Construction of $\mathcal{F}_1, \mathcal{F}_2, \ldots$!

X is reduced

Assume $\operatorname{depth} \mathcal{O}_{X,x} \geq 1$ ($\forall$ closed points x.

$$\begin{tikzcd} X & X \arrow[l, "\varphi = \text{Frobenius}"'] \arrow[d, "\pi"'] & \\ & \mathbb{P}^d_k & \underbrace{\mathbb{P}^1_k \times \cdots \times \mathbb{P}^1_k}_{d} \arrow[l, "\rho"'] \end{tikzcd}$$

finite + linear

$$\mathcal{F}_n := \varphi^n_*\Big(\pi^*\big(\rho_*(\mathcal{O}(p^n, 2p^n, \ldots, dp^n))\big)\Big)$$

An aside:

<u>Corollary</u> char $k = p$, k perfect, X CM projective k-scheme of dim d. The cone of cohomology tables for X coincides, "up to limits", with the cone of coh. tables of $\mathbb{P}^d_k$.

Def'n of χ_∞ and $[R]_{\cancel{\dim(R)}} \in G_0(R)$

$\mathrm{pd}_R M < \infty$, $\mathrm{length}_R M < \infty$

$$\chi_\infty(M) = \lim_{i \to \infty} \frac{\mathrm{length}_R\left(M \otimes_R {}^{\varphi^i}R\right)}{p^{i \cdot \dim(R)}}$$

Ex: $M = R/I$, $\chi_\infty(M) = e_{HK}(I)$

$\exists$ isom $\tau : G_0(R)_{\mathbb{Q}} \xrightarrow{\cong} \bigoplus_{i=0}^{\dim R} CH_i(R)_{\mathbb{Q}}$

$$[R]_{\dim(R)} = \tau^{-1}\left(\tau_{\dim(R)}[R]\right)$$

$$\tau = \tau_0 \oplus \cdots \oplus \tau_{\dim(R)}$$

$$[R]_{\dim(R)} \in G_0(R)_{\mathbb{Q}}$$

Very hard to compute. Sometimes

$[R]_{\dim(R)} = [R]$

Fact $\quad \mathrm{length}_R M < \infty \quad \mathrm{pd}_R M < \infty$

$$\chi_\infty(M) = \chi\left(\underline{F_\bullet} \otimes_R \underline{[R]_{\dim(R)}} \right)$$

$F_\bullet$ = minimal free resolution of M

Corollary [IMW] $\quad$ char $k = p$,
R is localization of a standard graded k-algebra at its hom. max'l ideal.
If $\mathrm{length}_R(M) < \infty$ + $\mathrm{pd}_R(M) < \infty$
then $\forall i$

$$\binom{d}{i} \chi_\infty(M) \geq \beta_i(M) \cdot e(R)$$

(Conj C)

R is not nec. CM

$F_\cdot : (0 \to F_d \to \cdots \to F_0 \to 0)$ $d = \dim R$

$\mathrm{length}\, H_i(F_\cdot) < \infty$ $\forall i$

$\leftarrow$ defined like $\chi_\infty(M)$

<u>Conj D</u> $\chi_\infty(F_\cdot) \geq e(R)$

Conj D has consequences for regular rings:

$Q = k[[x_1, \dots, x_d]]$

$\mathrm{length}_Q\, M < \infty$

$F_\cdot$ = min'l free res'n of M

Assume $\exists \ell$ $d(F_i) \leq m^\ell \cdot F_{i-1}$ $\forall i$

$R = k[[m^\ell]] = k[[$all monomials of degree at least $\ell]]$

$\subseteq Q$

If Conj. D holds for R, then $\text{length}_R(M) \geq \ell^d$.

<u>Note</u> This <u>is true</u> in the graded setting — consequence of "multiplicity conjecture" proven by Boij–Söderberg / Eisenbud–Schreyer

Thanks!!!

$$F_\bullet = \check{F}_\bullet \otimes_R Q$$

$\tilde{F}_\cdot$ = free complex / R

$$\chi_\infty^R(\hat{F}_\cdot) \underline{\geq} e(R)$$

$$\|$$

$$\chi^R(\tilde{\hat{F}}_\cdot)$$

$$\| \leftarrow [Q] = [R] \in G_0(R)_{\mathbb{Q}}$$

$$\chi^Q(F_\cdot)$$

$$\|$$

$$\text{length}_Q(M)$$

$R = k[[m^l]]$ $\quad e(R) = e(m^l, Q)$

$$= l^d$$

[Yhee] R does <u>not</u> have an Ulrich module