Saturation Bounds for Smooth Vars

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Plan: first 45 or 50 mins will discuss problem state results

• After break: discuss methods of pf.

Notation:

$$S = \mathbb{C}[x_0, \ldots, x_r]$$

$$S_+ = \mathfrak{m} = (x_0, \ldots, x_r) \text{ : irrel. max ideal.}$$

Intro: Macaulay's Thm:

Consider homog polys

$$f_0, \ldots, f_p \in S, \quad \deg f_i = d_i$$

$$d_0 \geq d_1 \geq \cdots \geq d_p.$$

$$J = (f_0, \ldots, f_p) \subseteq S$$

M's Thm. Assume J is $\mathfrak{m}$-primary, ie. $\dim_{\mathbb{C}}(S/J) < \infty$

(so $p \geq r$: main case $p = r$)

Then $J_t = S_t$ when

$$t \geq d_0 + \cdots + d_r - r.$$

Moreover, bound is (always) sharp when $p = r$.

Ex. Take $f_i = x_i^{d_i}$. Then

$$x_0^{d_0 - 1} \cdot \ldots \cdot x_r^{d_r - 1} \notin J.$$

I'm sure everyone here has their favorite pf of this Thm I'll indicate mine in second part of

What's maybe less familiar is a generalizn of this statement to powers

Generalization: with J as above, $(J^a)_t = S_t$ for

$$t \geq a d_0 + d_1 + \cdots + d_r - r.$$

Always sharp when $p = r$.

Ex. $f_i = x_i^{d_i}$. Then

$$x_0^{ad_0-1} \cdot x_1^{d_1-1} \cdot \ldots \cdot x_r^{d_r-1} \notin J^a.$$

If you're literal-minded and didn't know title of talk then you might raise objection that this is manifestly false. But point is we should interpret M's thm as statement about saturation

Ask: Analogues for more general J?

Slogan: If

$$X = \mathrm{Zeroes}(J) \subseteq \mathbb{P}^r$$

is a smooth variety, then exactly the same bounds hold.

Saturation –

Consider

$$J \subseteq S \quad \text{any homog ideal.}$$

Recall

$$J^{sat} = \{ f \mid \mathfrak{m}^N \cdot f \in J \text{ for } N \gg 0 \}$$
$$= (J : \mathfrak{m}^\infty)$$

Then

$$J^{sat}/J = H^0_{\mathfrak{m}}(S/J) \text{ vanishes in large degree,}$$

ie.

$$(J^{sat})_t = J_t \quad \text{for } t \gg 0$$

Def:

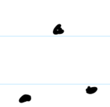
$$\mathrm{sat.deg}(J) = \text{least such } t.$$

Note: $J \subseteq S$ finite colength $\iff J^{sat} = S$. So

M's Thm $\iff$ computing sat deg of J, J^a.

Ex. Consider

$$J = (xy, yz, zx)^2 \subseteq \mathbb{C}[x,y,z]$$

(J = square of ideal of the three coord pts in $\mathbb{P}^2$. )

Then $J^{sat} = (xyz, J)$. $\square$

• Consider any var (or sch) $X \subseteq \mathbb{P}^r$. Let

$$I = I_X \subseteq S \text{ be homog ideal of } X.$$

Then I_X is saturated (ie $I_X^{sat} = I_X$) and if

$$J \subsetneq S \text{ is any ideal s.t. } \text{Zeroes}(J) = X \text{ (as scheme)}$$

then

$$J^{sat} = I_X.$$

and sat deg(J^a)

__Question:__ Given ideal J, can we bound sat. deg(J) in terms of the degs of its generators?

__Ex.__ Say

$$X = \text{hyperplane } (\ell = 0) \subseteq \mathbb{P}^r.$$

Take

$$f_i = x_i^{d-1}\ell$$

$$J = (x_0^{d-1}, \quad, x_r^{d-1}) \cdot (\ell)$$

presence of ℓ raises sat. deg by 1.

M's Thm $\Rightarrow$

$$\begin{aligned} \text{sat. deg } J &= (r+1)(d-1) - r + 1 \\ &= (r+1)d - 2r \end{aligned}$$

(Very close in shape to M's bound.)

__Ex__ (Caviglia) Take

$$J = (x^d, y^d, xz^{d-1} - yw^{d-1}).$$

Then

$$\text{sat deg}(J) \approx d^2. \quad \square$$

Our first main result says that the first example is typical for ideals defining smooth varieties.

Thm A. As before, consider
$$J = (f_0, \dots, f_p).$$
where
$$\deg f_i = d_i \quad , \quad d_0 \geq d_1 \geq \dots \geq d_p$$
Assume:
$$X = \mathrm{Zeroes}(J) \text{ is smooth proj var } / \mathbb{C}.$$
Then
$$J_t = (I_X)_t \quad \text{for } t \geq d_0 + \dots + d_r - r.$$

Rmk: When $J = (x_0^{d-1}, \ \ , x_r^{d-1}) \cdot \ell$, Thm gives
$$\text{sat. deg}(J) \leq (r+1)d - r,$$
whereas actual sat. deg is $(r+1)d - 2r$. So asymptotically sharp.

Powers –

Recall that if $Z \subseteq \mathbb{P}^r$ is any reduced var w. homog ideal I_Z, then
$$I_Z^{(a)} = \{ f \in S \mid \underset{x \in Z}{\mathrm{ord}_x f \geq a} \text{ at general (or all)} \}$$
symbolic power of I_Z

Prop: If $X \subseteq \mathbb{P}^r$ is non-sing, then
$$I_X^{(a)} = (I_X^a)^{sat}.$$
Esp
$$(I_X^{(a)})_t = (I_X^a)_t \quad \text{for } t \geq \text{sat. deg}(I_X^a).$$

Ex. $X = \{ \therefore \subseteq \mathbb{P}^2 \}$, $I_X = (xy, yz, zx)$. Then
$$I_X^{(a)} = I^a + (xyz) I_X^{a-2} + (xyz)^2 \cdot I_X^{a-4} + \dots$$
So
$$\text{sat. deg}(I_X^a) = 2a.$$

Thm A$'$: Take J as in Thm A, so

$$\text{Zeroes}(J) = \text{sm var} \subseteq \mathbb{P}^r.$$

Then

$$\text{sat.deg}\, J^a \leq ad_0 + d_1 + \cdots + d_r - r.$$

Esp, if I_X has gens of degs $d_0 \geq d_1 \geq \cdots \geq d_m$, then

$$(I_X^{(a)})_t = (I_X^a)_t \quad \text{for } t \geq ad_0 + \cdots + d_r - r.$$

Ex. $X = (\because \subseteq \mathbb{P}^2)$. Thm gives

$$\text{sat. deg}(I_X^a) \leq 2a + 2,$$

off by 2.

Saturation and regularity -

. Consider

$$X \subseteq \mathbb{P}^r \;\; \text{w ideal sheaf } \mathcal{I}_X \subseteq \mathcal{O}_{\mathbb{P}^r}$$

Recall: say X is m-regular if

$$H^i(\mathbb{P}^r, \mathcal{I}_X(m-i)) = 0 \quad i > 0$$

$$\Updownarrow$$

I_X being m-regular in sense of Eisenbud-Goto

We will call this "arithmetic regularity"

Note: For arb ideal $J \subseteq S$

arithmetic reg of J

$$\updownarrow$$

- regularity of J^{sat} (or $\mathcal{J} = \widetilde{J} \subseteq \mathcal{O}_{\mathbb{P}^r}$)
- Bound on sat. deg J.

Geramita – Gimigliano – Pittoud, Sidman, Chandler – If

$X \subseteq \mathbb{P}^r$ is finite set, and X is m-reg,

then

$$\text{sat. deg}(I_X^a) \leq m \cdot a.$$

Ex. $X = \{ \therefore \} \subseteq \mathbb{P}^2$: 2-reg, and

$$\text{sat deg}(I_X^a) = 2a.$$

Ask: What about $\dim X \geq 1$?

Thm B. Same statement holds when $X \subseteq \mathbb{P}^r$ a smooth curve
ie

$$\text{sat. deg}(I_X^a) \leq a \cdot \text{reg}(X).$$

(I think: X reduced ok.)

Ex. Thm optimal eg when $X \subseteq \mathbb{P}^4$ is rat normal curve.

Question: Is it reasonable to expect analogous statement when $X \subseteq \mathbb{P}^r$ smooth of $\dim \geq 2$?

Inputs to Pfs –

- Inspired by paper of Cooper + 9, who used symm products of Hilbert Burch resolns to study symbolic powers of codim 2 CM ideals.

- To illustrate, I'll explain how their methods would apply to pts in $\mathbb{P}^2$.

Ex. Consider m-reg. finite set

$$X \subseteq \mathbb{P}^2.$$

We'll use ideas of [C+9] to prove sat. deg $(I_X^a) \leq am$.

(1°) Use min gens of I_X to construct resoln

$$0 \longrightarrow U_1 \longrightarrow U_0 \xrightarrow{\varepsilon} \mathcal{I}_X \longrightarrow 0$$

w.

$$U_0 = \oplus \mathcal{O}_{\mathbb{P}^2}(-b_i) \quad \text{all } b_i \leq m$$

$$U_1 = \oplus \mathcal{O}_{\mathbb{P}^2}(-c_j) \quad \text{''} \quad c_j \leq m+1$$

(2°). Consider

$$S^a U_0 \xrightarrow{S^a \varepsilon} \mathcal{I}_X^a \longrightarrow 0 \qquad (*)$$

$$\text{Image of } H^0(\mathbb{P}^2, S^a U_0 \otimes \mathcal{O}_{\mathbb{P}}(t)) \longrightarrow H^0(\mathbb{P}^2, \mathcal{I}_X^a \otimes \mathcal{O}_{\mathbb{P}}(t))$$

is

$$(I_X^a)_t \subseteq (I_X^{(a)})_t.$$

(3°). $(*)$ sits in long exact seq

$$0 \leftarrow \mathcal{I}_X^a \longleftarrow \underset{\substack{\uparrow \\ a \cdot m\text{-reg}}}{S^a U_0} \longleftarrow \underset{\substack{\uparrow \\ (am+1)\text{-reg}}}{S^{a-1} U_0 \otimes U_1} \longleftarrow \underset{\substack{\uparrow \\ (am+2)\text{-reg}}}{S^{a-2} U_0 \otimes \Lambda^2 U_1} \longleftarrow \cdots$$

(4°). Result follows from:

<u>Lemma</u>. Consider exact seq of sheaves on $\mathbb{P}^n$:

$$\cdots \longrightarrow E_2 \longrightarrow E_1 \longrightarrow E_0 \longrightarrow \mathcal{F} \longrightarrow 0$$

Assume E_i is $(l+i)$-reg. Then

$$H^0(\mathbb{P}, E_0(t)) \twoheadrightarrow H^0(\mathbb{P}, \mathcal{F}(t)) \quad \text{for } t \geq l. \quad ▨$$

For Thm B, will want to extend to X of higher dim.

- Consider m-reg sm. var $X \subseteq \mathbb{P}^r$
- $\mathcal{I}_X$ has loc free resoln:

$$0 \leftarrow \mathcal{I}_X \longleftarrow U_0 \longleftarrow U_1 \longleftarrow U_2 \longleftarrow \cdots \qquad (*)$$

$$\operatorname{reg}(U_i) \leq m+i,$$

- Weyman: $S^a(*)$ has shape

$$0 \leftarrow \mathcal{I}_X^a \longleftarrow \underset{\substack{\uparrow \\ am\text{-reg}}}{S^a U_0} \longleftarrow \underset{\substack{\uparrow \\ (am+1)\text{ reg}}}{S^{a-1} U_0 \otimes U_1} \longleftarrow \underset{\substack{\uparrow \\ (am+2)\text{ reg}}}{\begin{matrix} S^{a-2} U_0 \otimes \Lambda^2 U_1 \\ \oplus \\ S^{a-1} U_0 \otimes U_2 \end{matrix}} \longleftarrow$$

- $S^a(*)$ not exact, but is exact off X
- Check à la [GLP] that if $\dim X \leq 1$, still get

$$H^0(S^a U_0(t)) \twoheadrightarrow H^0(\mathcal{I}_X^a(t)) \qquad \text{for } t \geq am.$$

Powers of Koszul Complex –

- The Macaulay-type statements come from Koszul cx and its powers.
- To warm up, let me recall one way of proving classical statement

Ex ("Mac Classic") Take $f_0, \ldots, f_r \in S$ gen $\mathfrak{m}$-prim $J \subseteq S$.
Let

$$U = \bigoplus_{i=0}^{r} \mathcal{O}_{\mathbb{P}^r}(-d_i)$$

$$\varepsilon : U \longrightarrow \mathcal{O}_{\mathbb{P}^r} \longrightarrow 0 \qquad \text{map given by } f_i$$

(surj as map on sheaves since J is $\mathfrak{m}$-primary.)

So
$$J = \mathrm{im}\left(H^0_*(U) \longrightarrow H^0_*(\mathcal{O}_{\mathbb{P}}) = S \right)$$

· Now form Kosz cx from ε:
$$0 \to \Lambda^{r+1} U \longrightarrow \qquad \longrightarrow \Lambda^2 U \longrightarrow U \longrightarrow \mathcal{O} \longrightarrow 0$$
$$\mathrm{reg}(\Lambda^i U) = d_0 + \quad + d_{i-1} \leq (d_0 + .. + d_r - r) + i$$

· Lemma $\Rightarrow$
$$H^0(U(t)) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}}(t)) \qquad t \geq \Sigma d_i - r \qquad \square$$

What about powers?

<u>Def.</u> Let U = v.b on sm var M. Given $a \geq 1$, define

$$S^{a,1^{k-1}}(U) = \left\{ \begin{array}{l} \text{Schur power of } U \text{ corresp} \\ \text{to partition } (a, 1, ., 1) \end{array} \right\}$$
$$= \ker\left(S^a U \otimes \Lambda^{k-1} U \longrightarrow S^{a+1} U \otimes \Lambda^{k-2} U \right)$$
$$= \mathrm{im}\left(S^{a-1} U \otimes \Lambda^k U \longrightarrow S^a U \otimes \Lambda^{k-1} U \right)$$

<u>Rmks.</u> · When $a = 1$, $S^{a,1^{k-1}} U = \Lambda^k U$.

· $S^{a,1^{k-1}} U = 0$ when $k > \mathrm{rk}\, U$.

· If $U = \bigoplus_{i=0}^{m} \mathcal{O}_{\mathbb{P}^r}(-d_i)$ w. $d_0 \geq d_1 \geq \cdots$, then
$$\mathrm{reg}(S^{a,1^{k-1}} U) = a d_0 + \cdots + d_{k-1}.$$

<u>Prop.</u> (Srinivasan..) Consider map of vbs

$$\varepsilon : \mathcal{U} \longrightarrow \mathcal{O}_M$$

For $a \geq 1$, ε determines cx:

$$\cdots \longrightarrow S^{a,1^2}\mathcal{U} \longrightarrow S^{a,1}\mathcal{U} \longrightarrow S^a\mathcal{U} \xrightarrow{S^a(\varepsilon)} \mathcal{O}_M \qquad (*)_a$$

This is acyclic under same assumptions as for Kosz. cx $(*)_1$.

- Prop $\Rightarrow$ Variant of M's thm for J^a.
- Prop $\Rightarrow$ Zariski's thm that if $Z \subseteq \mathbb{P}^r$ is compl. intersection of dim ≥ 0, then

$$I_Z^a \text{ saturated } \forall\, a \geq 1$$

<u>Idea of Pf of Thm A</u>

- As above, let $\mathcal{U} = \bigoplus_{i=0}^{p} \mathcal{O}_{\mathbb{P}^r}(-d_i)$.
- We assume given surj

$$\varepsilon : \mathcal{U} \twoheadrightarrow \mathfrak{d}_X \subseteq \mathcal{O}_{\mathbb{P}^r}, \qquad X \subseteq \mathbb{P}^r \text{ sm.}$$

- Want to apply constr above to $S^a\mathcal{U} \to \mathfrak{d}_X^a$, but resulting complex not exact.
- Pass to blow-up:

$$\mu : \mathbb{P}' = \mathrm{Bl}_X(\mathbb{P}^r) \longrightarrow \mathbb{P}^r$$

$$\mathbb{P}' \supseteq \mathbb{E} = \text{exceptnl div.} \qquad \mathfrak{d}_X \cdot \mathcal{O}_{\mathbb{P}'} = \mathcal{O}_{\mathbb{P}'}(-\mathbb{E}).$$

- Let $\mathcal{U}' = \mu^*\mathcal{U}$. Now get surj map

$$\varepsilon' : \mathcal{U}' \twoheadrightarrow \mathcal{O}_{\mathbb{P}'}(-\mathbb{E})$$

So on $\mathbb{P}'$ get exact complex:

$$\cdots \longrightarrow S^{a,1}\mathcal{U}' \otimes \mathcal{O}(\mathbb{E}) \longrightarrow S^a\mathcal{U}' \xrightarrow{S^a\varepsilon'} \mathcal{O}_{\mathbb{P}'}(-a\mathbb{E}) \longrightarrow 0$$

· As before, issue is to show that suitable twist of $S^a\varepsilon'$ surj on H^0.

<u>Difficulty</u>: For $k \geq e = \mathrm{codim}(X)$, twists of $\mathcal{O}_{\mathbb{P}}(k\mathbb{E})$ have non-vanish. cohom in intermediate degrees.

So need to compute. Two essential pts:

(A). $$R^i\mu_*\mathcal{O}_{\mathbb{P}'}(k\mathbb{E}) = \mathcal{E}xt^i_{\mathcal{O}_{\mathbb{P}}}(\mathcal{I}_X^{k+1-e}, \mathcal{O}_{\mathbb{P}^r})$$

(B). <u>Thm</u> Let $X \subseteq \mathbb{P}^r$ be sm proj var $/\mathbb{C}$ w normal bundle
$$N = N_{X/\mathbb{P}}.$$

Then:
$$H^i(X, S^kN \otimes \det N \otimes \mathcal{O}_X(\ell - r)) = 0 \quad \text{for } i > 0$$

for all $k, \ell \geq 0$.

(This reduces to Kodaira–Nakano vanishing on X.)