Normal Reduction Numbers, Normal Hilbert coefficients and Elliptic ideals in normal 2-dimensional local domains

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Normal Reduction Numbers $nr(I)$ and $\bar{r}(I)$

In this talk, our $(A, \mathfrak{m})$ is an excellent two-dimensional normal local domain containing the field $k \cong A/\mathfrak{m}$ or a graded ring $A = \oplus_{n \geq 0} A_n$ with $A_0 = k$ and we assume k is algebraically closed.
Our ideal I is always an integrally closed $\mathfrak{m}$-primary ideal and Q be a minimal reduction of I (a parameter ideal with $I^{r+1} = QI^r$ for some $r \geq 1$).
Then we define normal reduction numbers as
$\mathrm{nr}(I) = \min\{n \mid \overline{I^{n+1}} = Q\overline{I^n}\}$, $\bar{r}(I) = \min\{n \mid \overline{I^{N+1}} = Q\overline{I^N}, \forall N \geq n\}$.
$\mathrm{nr}(A) = \max\{\mathrm{nr}(I) \mid I \subset A\}$ and $\bar{\mathrm{r}}(A) = \sup\{\bar{\mathrm{r}}(I) \mid I \subset A\}$

Also the normal Hilbert coefficients $\bar{e}_i(I)$ $(i = 0, 1, 2)$ are defined by

$$\ell_A(A/\overline{I^{n+1}}) = \bar{e}_0(I)\binom{n+2}{2} - \bar{e}_1(I)\binom{n+1}{1} + \bar{e}_2(I) := P_I(n)$$

for $n \gg 0$. Normal Hilbert coefficients are studied by Huneke ('87), Itoh ('88,'92), Corso-Polini-Rossi ('05), ...
Our aim is to know the behavior of these invariants for every integrally closed $\mathfrak{m}$ primary ideal I of a given ring A.

Resolution of singularities, $I = I_Z$

For an ideal (always $\mathfrak{m}$ primary and integrally closed) we can choose a resolution

$$f : X \to \operatorname{Spec}(A), \text{ so that } I\mathcal{O}_X = \mathcal{O}_X(-Z), \text{ and } I = \mathrm{H}^0(X, \mathcal{O}_X(-Z))$$

where $Z = \sum_{i=1}^r n_i E_i$, where

$$\mathbb{E} := f^{-1}(\mathfrak{m}) = \bigcup_{i=1}^r E_i.$$

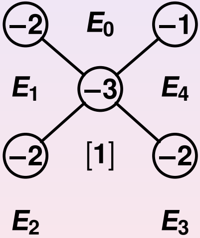
We write $I = I_Z$ in this case and we say I is represented on X.
In other word, each valuation v_{E_i} is a Rees valuation of I so that for $a \in A$,

$$a \in I \iff v_{E_i}(a) \geq n_i \ (1 \leq i \leq r).$$

and for $Q \subset I$, Q is a reduction of I if and only if $Q\mathcal{O}_X = I\mathcal{O}_X$.

Dual graph and intersection matrix

We can represent curves of $\mathbb{E}$ by the dual graph. The following graph is a resolution of $k[x, y, z]/(x^3 + y^4 + z^6)$, where [1] means that the genus of E_0 is 1 and the intersection matrix $\mathbb{M} = (E_i E_j)_{i=0}^4$.



$$\mathbb{M} = \begin{pmatrix} & E_0 & E_1 & E_2 & E_3 & E_4 \\ E_0 & -3 & 1 & 1 & 1 & 1 \\ E_1 & 1 & -2 & 0 & 0 & 0 \\ E_2 & 1 & 0 & -2 & 0 & 0 \\ E_3 & 1 & 0 & 0 & -2 & 0 \\ E_4 & 1 & 0 & 0 & 0 & -1 \end{pmatrix}$$

In this case

$\mathfrak{m} = (x, y, z) = I_Z$, with $Z = 2E_0 + (E_1 + E_2 + E_3) + 3E_4$ and one can compute $e_0(\mathfrak{m}) = -Z^2 = 3$. The Gorenstein condition of A is equivalent to say that the canonical divisor is represented by a cycle. and in this example $K_X = -4E_0 - 2(E_1 + E_2 + E_3) - 3E_4$ and we have a "Riemann-Roch" formula due to M. Kato to compute $\ell_A(A/I_Z)$ and then we will recall that $K_X Z = 3$.

Cohomology Groups and arithmetic genus

We denote $\ell_A(H^1(X,\mathcal{F})) = h^1(X,\mathcal{F})$. When X is clear from the context, we write simply $h^1(\mathcal{F})$. We introduce important invariants of A and I. Since $\dim A = 2$, $\mathrm{H}^2(X,\mathcal{F}) = 0$ for every coherent $\mathcal{F}$. Let $f : X \to \operatorname{Spec}(A)$ be a resolution.

Definition 2.1

1. $p_g(A) = h^1(\mathcal{O}_X)$
2. (Artin) A is **rational** if $p_g(A) = 0$.
3. (S.S.T.Yau) A is strongly elliptic if $p_g(A) = 1$.
4. $q(I_Z) = h^1(\mathcal{O}_X(-Z))$ and $q(nI_Z) = q(\overline{I^n}) = h^1(\mathcal{O}_X(-nZ))$.

Remark 2.2

We are also interested to express these invariants by language of commutative algebra. For a parameter ideal $Q \subset \mathfrak{m}^N$ for large N, we can write using tight closure; $p_g(A) = \ell_A(Q^*/Q)$. How about $q(I)$?
If $A = \oplus_{n\geq 0} A_n$ is graded, then $p_g(A) = \sum_{n\geq 0} \ell_A(\mathrm{H}^2_{\mathfrak{m}}(A)_n)$.

Elliptic Singularity; Arithmetic genus

We say $Z \in \sum_{i=1}^{r} \mathbb{Z}E_i$ is anti-nef if $ZE_i \leq 0$ for every i. Since the intersection matrix (E_iE_j) is negative definite, $Z > 0$ if $Z \neq 0$ is anti-nef.

Definition 2.3

(1) We call the unique minimal anti-nef cycle the fundamental cycle of X and denote $\mathbb{Z}_X$.
(2) [Wagreich] A is called an elliptic singularity if $p_a(\mathbb{Z}_X) = 1$, where $p_a(Z)$ is defined by $2(p_a(Z) - 1) = Z^2 + ZK_X$.

The notion of elliptic singularity is very important in this talk. But we do not have algebraic method to determine elliptic singularities.

Example 2.4 (OWY4)

Let $A = k[x, y, z]/(x^a + y^b + z^c)$ with $2 \leq a \leq b \leq c$. Then

1. A is a rational singularity if and only if $1/a + 1/b + 1/c > 1$.
2. $\bar{r}(\mathfrak{m}) \leq 2$ if and only if $\lfloor \frac{(a-1)b}{a} \rfloor \leq 2$. We can assure that either A is elliptic or $\bar{r}(A) \geq 3$ except the case $(a, b, c) = (3, 4, 6)$ and $(3, 4, 7)$.

Main Results

Let us list our main results. Before that we define;

Proposition-Definition 2.5

For every I, we have $q(I) \leq p_g(A)$ and we call I a p_g -ideal if $q(I) = p_g(A)$.

Theorem 2.6

1. $\bar{r}(I) = 1$ *if and only if I is a p_g -ideal. Every ring A has a p_g-ideal.*
2. A *is a rational singularity if and only if* $\bar{r}(A) = 1$.
3. *(Okuma) If A is an elliptic singularity, then* $\bar{r}(A) = 2$.
4. *If* $A = k[[x, y, z]]/(f)$ *is a homogeneous hypersurface of degree d, then* $\bar{r}(A) = nr(\mathfrak{m}) = d - 1$.
5. *For any $n \geq 1$, there exist A and ideal I in A with* $nr(I) = 1$ *and* $\bar{r}(I) = n = p_g(A) + 1$.

Question 2.7

If $\bar{r}(A) = 2$, then is A an elliptic singularity ?

$q(nI)$ is non-increasing

In the following, we fix a resolution X such that $I\mathcal{O}_X = \mathcal{O}_X(-Z)$.

Lemma 3.1

We call $\mathcal{O}_X(-Z)$ has no fixed compoments if $\mathrm{H}^0(X, \mathcal{O}_X(-Z - E_i)) \subsetneq \mathrm{H}^0(X, \mathcal{O}_X(-Z))$ for every E_i. If $\mathcal{O}_X(-Z)$ has no fixed compoments, then $h^1(\mathcal{O}_X(-Z - Z')) \leq h^1(\mathcal{O}_X(-Z'))$ for any cycle Z' and in particular, $h^1(\mathcal{O}_X(-Z)) \leq p_g(A)$.

Proof.

Let s be a general element of $\mathrm{H}^0(X, \mathcal{O}_X(-Z))$. Since $\mathcal{O}_X(-Z)$ has no fixed compoments, $\mathcal{C} := \mathrm{Coker}[\mathcal{O}_X \xrightarrow{s} \mathcal{O}_X(-Z)]$ has support finite over $\mathrm{Spec}(A/sA)$ and hence is affine. Then, taking the cohomology long exact sequence of the excat sequence

$$0 \to \mathcal{O}_X(-Z') \to \mathcal{O}_X(-Z - Z') \to \mathcal{C} \otimes \mathcal{O}_X(-Z) \to 0$$

we have a surjection $\mathrm{H}^1(\mathcal{O}_X(-Z')) \to \mathrm{H}^1(\mathcal{O}_X(-Z - Z'))$ since $\mathrm{H}^1(\mathcal{C} \otimes \mathcal{O}_X(-Z)) = 0$ having affine support. □

Values of $q(I)$

We showed that $0 \leq q(I) \leq p_g(A)$ and for long years we expected that for a given ring A and for every $0 \leq q \leq p_g(A)$, there is an ideal I of A with $q(I) = q$. It is only one week ago that we could prove that is not the case.

Example 3.2

Let $A = k[[x, y, z]]/(f)$ where f is homogeneous of degree d.

1. If $d = 4$ and $0 \leq q \leq 4 = p_g(A)$, $q \neq 2$, then there exists I such that $q(I) = q$. There is no I with $q(I) = 2$.
2. If $d = 5$ and $0 \leq q \leq 10 = p_g(A)$, $q \neq 5$, then there exists I such that $q(I) = q$. There is no I with $q(I) = 5$.

Proof is geometric using vanishing theorem of H^1.
We can also make such examples for a standard graded ring of high degree whose Proj is not a hyperelliptic curve. (A smooth curve C is hyperelliptic if there is a $2 : 1$ map from C to $\mathbb{P}^1$.)

Lipman's exact sequence

Let $Q = (a, b)$ be a minimal reduction of I. Since $a, b \in \mathrm{H}^0(X, \mathcal{O}_X(-Z))$, multiplication by (a, b) induces $\mathcal{O}_X \to \mathcal{O}_X(-Z)^{\oplus 2}$ and the following exact sequence due to Lipman (1969).

$$0 \to \mathcal{O}_X \overset{(a,b)}{\to} \mathcal{O}_X(-Z)^{\oplus 2} \overset{\begin{pmatrix} -b \\ a \end{pmatrix}}{\longrightarrow} \mathcal{O}_X(-2Z) \to 0.$$

Tensoring $\mathcal{O}_X(-(n-1)Z)$ and taking the cohomology long exact sequence,

$$\mathrm{H}^0(\mathcal{O}_X(-(nZ)^{\oplus 2}) = \overline{I^n}^{\oplus 2} \overset{\alpha}{\to} \mathrm{H}^0(\mathcal{O}_X(-(n+1)Z)) = \overline{I^{n+1}}$$

$$\to \mathrm{H}^1(\mathcal{O}_X(-(n-1)Z) \to \mathrm{H}^1(\mathcal{O}_X(-nZ)^{\oplus 2}) \to \mathrm{H}^1(\mathcal{O}_X(-)n+1)Z)) \to 0$$

Since the image of $\alpha : \overline{I^n}^{\oplus 2} \to \overline{I^{n+1}}$ is $Q\overline{I^n}$, we have;

Corollary 3.3

1. *If* $q(nI) = q((n+1)I)$ *for some* $n \geq 0$*, then* $q((n+1)I) = q((n+2)I) = \ldots := q(\infty I)$*. Hence for* $n \geq p_g(A)$*, we have* $q(nI) = q(\infty I)$*.*
2. *For all* n*,* $\ell_A(\overline{I^{n+1}}/Q\overline{I^n}) = 2q(nI) - [q((n-1)I) + q((n+1))I]$*.*

$nr(I), \bar{r}(I)$ and $q(nI)$

From Lipman's exact sequence and Lemma 3.1, we have also;

Corollary 3.4

Let I be an integrally closed ideal and Q be its minimal reduction. Then we have;

1. *I is a p_g - ideal if and only if $\bar{r}(I) = 1$.*
2. *$nr(I) = \min\{n \mid 2q(nI) = q((n-1)I) + q((n+1)I)\}$ and* *$\bar{r}(I) = \min\{n \mid q((n-1)I) = q(nI)\}$.*
3. *$\bar{r}(I) \leq p_g(A) + 1$ and if we have equality, then $nr(I) = 1$.*
4. *(OWY4) If $nr(I) = r$ for some I, then $p_g(A) \geq \binom{r}{2} + q(rI)$.*

p_g ideals exist abundantly.

Theorem 3.5

(OWY3) Let I be an integrally closed ideal and $f \in I$ is a general element. Then there exists a p_g ideal I' satisfying the conditions $I' \subset I$ and $f \in I'$.

(4) and (5) of Main Results

Theorem 3.6 (OWY5)

Assume A has a resolution $f : X_0 \to \mathrm{Spec}(A)$, where $\mathbb{E}$ is a single smooth curve F of genus $g > 0$ and $-F^2 = d$. Let $I = I_Z$ be represented on X and E_0 be the curve on X of genus g. If $\mathbf{gon}(\mathbf{F})$ is minimal of degree of divisor D on F with $h^0(D) \geq 2$, we have

1. *If $ZE_0 = 0$, then $\bar{\mathrm{r}}(I) \leq \lfloor (2g-2)/d \rfloor + 2$*
2. *If $ZE_0 < 0$, then $\bar{\mathrm{r}}(I) \leq \lfloor (2g-2)/\mathbf{gon}(\mathbf{F}) \rfloor + \mathbf{1}$.*

The important point of our proof is the Vanishing Thorem of Röhr ('95).
The following example is very much different from the previous one.

Example 3.7 (OWY5)

Let $R = k[X, Y, Z]/(X^2 + Y^{2g+2} + Z^{2g+2})$, which we consider a graded ring putting $\deg(X) = g + 1$ and $\deg(Y) = \deg(Z) = 1$. $A = R^{(g)}$, the g-th Veronese subring of R so that $A = k[y^g, y^{g-1}z, \ldots, z^g, xy^{g-1}, \ldots, xz^{g-1}]$. Put $I = (y^g, y^{g-1}z, A_{\geq 2})$. Then we can show that $nr(I) = 1$ and $\bar{r}(I) = g + 1 = p_g(A) + 1$.

Elliptic Ideals, Strongly Elliptic Ideals

We would propose the following definition.

Definition 3.9

Let I be an integrally closed $\mathfrak{m}$ primary ideal I of A.

1. I is called an **elliptic ideal** if $\bar{r}(I) = 2$.
2. I is called a **strongly elliptic ideal** if $\bar{r}(I) = 2$ and $q(I) = p_g(A) - 1$, or equivalently, $\bar{e}_2(I) = 1$.

The following statement is obvious from the definition.

Example 3.10

(1) If A is an elliptic singularity then every integrally closed ideal of A is either a p_g- ideal or elliptic ideal.
(2) A is strongly elliptic, if and only if every integrally closed ideal I of A is either p_g -ideal or strongly elliptic.
(3) If $q(I) = 0$, then I is elliptic. Every A has I with $q(I) = 0$ since $\mathrm{H}^1(\mathcal{O}_X(-Z)) = 0$ if $-Z$ is sufficiently ample.

Normal Hilbert function via Riemann-Roch Theorem

Theorem 3.11 (Kato's Riemann-Roch Theorem)

If $I = I_Z$*, then we have* $\ell_A(A/I) = \dfrac{-Z^2 - K_X Z}{2} + [p_g(A) - q(I_Z)]$.

By this Theorem, we can write normal Hilbert function using Z, K_X.

Corollary 3.12

$$\ell_A(A/\overline{I^{n+1}}) = -Z^2\binom{n+2}{2} - \frac{-Z^2 + ZK_X}{2}\binom{n+1}{1} + [p_g(A) - q((n+1)I)]$$

Hence we have $e_0(I) = -Z^2, \bar{e}_1(I) = \dfrac{-Z^2 + K_X}{2}, \bar{e}_2(I) = p_g(A) - q(\infty I)$ *and* $\ell_A(A/\overline{I^{n+1}}) = P_I(n)$ *holds if* $q((n+1)I) = q(\infty I)$ *or* $n \geq \bar{r}(I) - 1$.

In slide 6 we showed a resolution of $A = k[x, y, z]/(x^3 + y^4 + z^6)$ and for $\mathfrak{m} = I_Z, Z^2 = -3$ and $K_X Z = 3$. Then we have $1 = \ell(A/\mathfrak{m}) = [p_g(A) - q(I_Z)]$ and since $p_a(A) = 2$ we have $q(I_Z) = 2$.

The case of p_g - ideals

Recall normal Rees algebra and normal associated graded ring of I.

$$\overline{\mathcal{R}}(I) = \oplus_{n\geq 0}\overline{I^n}t^n \text{ and } \overline{G}(I) = \oplus_{n\geq 0}\overline{I^n}/\overline{I^{n+1}}t^n$$

Theorem 4.1 (OWY2)

The following conditions are equivalent for I.

1. I *is a* p_g*-ideal.*
2. $\bar{\mathrm{r}}(I) = 1$.
3. $\bar{e}_1(I) = e_0(I) - \ell_A(A/I)$.
4. $\bar{e}_2(I) = 0$.
5. $\overline{\mathcal{R}}(I)$ *is Cohen-Macauly.*
6. $\overline{G}(I)$ *is CM with* $a(\overline{G}(I)) < 0$.

Characterizations of Elliptic Ideals.

Theorem 4.2

The following conditions are equivalent:

1. I *is an elliptic ideal. Namely,* $p_g(A) > q(I) = q(\infty I)$.
2. $\bar{e}_1(I) = e_0(I) - \ell_A(A/I) + \bar{e}_2(I)$ *and* $\bar{e}_2(I) > 0$.
3. $\ell_A(A/\overline{I^{n+1}}) = \bar{P}_I(n)$ *for all* $n \geq 0$ *and* $\bar{e}_2(I) > 0$.
4. $\bar{G}$ *is Cohen-Macaulay with* $a(\bar{G}) = 0$.

When this is the case, $\ell_A([H^2_{\mathfrak{M}}(\bar{G})]_0) = \ell_A(\overline{I^2}/QI) = \bar{e}_2(I)$. *If, moreover,* $\mathfrak{m}\overline{I^2} \subset QI$, *then* $\overline{\mathcal{R}}(I)$ *is a Buchsbaum ring.*

There are also abundant elliptic ideals in a given ring A.

Proposition 4.3

For any I, $\overline{I^n}$ *is an elliptic ideal if* $n \geq \bar{\mathrm{r}}(I) - 1$.

Characterizations of Strongly Elliptic Ideals.

Strongly elliptic ideals are characterized as follows.

Theorem 4.4

Then the following conditions are equivalent:

1. $\bar{r}(I) = 2$ *and* $\ell_A(\overline{I^2}/QI) = 1$*; That is,* I *is strongly elliptic.*
2. $q(I) = q(\infty I) = p_g(A) - 1$.
3. $\bar{e}_2(I) = 1$.
4. $\bar{e}_1(I) = e_0(I) - \ell_A(A/I) + 1$ *and* $\mathrm{nr}(I) = \bar{\mathrm{r}}(I)$.
5. $\bar{G}$ *is Cohen-Macaulay with* $a(\bar{G}) = 0$ *and* $\ell_A([H^2_{\mathfrak{M}}(\bar{G})]_0) = 1$.

In this case, $\bar{\mathcal{R}}$ *is a Buchsbaum ring with* $\ell_A(H^2_{\mathfrak{M}}(\bar{\mathcal{R}})) = 1$.

Elliptic ideal is strongly elliptic in some cases;

Proposition 4.5

If $(A, \mathfrak{m})$ *is Gorenstein and if* $\mathfrak{m}$ *is elliptic, then* $\mathfrak{m}$ *is strongly elliptic.*

Existence of Strongly Elliptic Ideals

For strongly elliptic ideals, we have positive answer for elliptic singularities.

Proposition 5.1

If A is elliptic, then for any $q, 0 \leq q \leq p_g(A)$, there exists an integrally closed ideal I with $q(I) = q$. In particular, A has a strongly elliptic ideal.

But on the other hand, some A does not have any strongly elliptic ideals. Since $q(\infty I) = p_g(A) - 1$ if $I = I_Z$ is strongly elliptic, then if we put $Y_I = \mathrm{Proj}(\mathcal{R}_A(I))$, the normal blowing-up of I, then we have $h^1(\mathcal{O}_Y) = 1$ and there is a cycle C such that $ZC = 0$ and $h^1(\mathcal{O}_C) = p_g(A) - 1$.

Example 5.2

Let $A = \oplus_{n \geq 0} \mathrm{H}^0(C, \mathcal{O}_C(nD))$, where C is a smooth curve of genus $g \geq 2$ and D is a divisor on C with $\deg D \geq 2g - 1$ (or $\mathrm{Spec}(A)$ can be resolved with unique exceptional curve C with $-C^2 \geq 2g - 1$). Then A has no strongly elliptic ideals, since for any cycle C on any resolution X of $\mathrm{Spec}(A)$, $h^1(\mathcal{O}_C)$ is either 0 or $g = p_g(A)$.

Normal and Non-normal Strongly Elliptic Ideals

If I is strongly elliptic, then since $\ell_A(\overline{I^2}/QI) = 1$ and $QI \subset I^2 \subset \overline{I^2}$, either $I^2 = QI$ or I^n are integrally closed for all $n \geq 2$.
If A is elliptic, we can distinguish these 2 cases by certain intersection number.

Theorem 5.3

If A is strongly elliptic, $I = I_Z$ be an elliptic ideal represented on X and D be the minimally elliptic cycle on X. Then I^2 is integrally closed if and only if $-DZ \geq 3$.

Remark 5.4

If I is strongly elliptic, then $I^2 \subset \mathrm{core}(I)$ if I is not normal and $\mathfrak{m}\overline{I^2} \subset \mathrm{core}(I)$ if I is normal.

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Thank you very much !

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