Betti numbers of monomial ideals fixed by permutation of the variables

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First Part: Motivation, example, basic properties

Second Part: Results

Main target ① $\mathfrak{S}_n$-invariant monomial ideals

$\mathfrak{S}_n$: nth symmetric group

$R_n := k[x_1,\dots,x_n] \curvearrowleft \mathfrak{S}_n$ (permutations of the variables) (k: field, $\deg(x_i)=1$)

A (monomial) ideal $I \subset R_n$ is $\mathfrak{S}_n$-invariant $\Leftrightarrow$ $\sigma(I)=I$ $\forall \sigma \in \mathfrak{S}_n$

Ex (n=2)

$\mathfrak{S}_n$-invariant	not $\mathfrak{S}_n$-invariant
(x_1^3, x_2^3)	(x_1^3, x_1x_2)
$(x_1^5x_2, x_1x_2^5, x_1^2x_2^2)$	(x_1^3, x_2^2)

Main target ② Betti numbers・Betti tables

$\beta_{ij}(I) = \dim_k \mathrm{Tor}_i(I,k)_j$ (i,j)th graded Betti number of I

These numbers appear in the minimal free resolutions of $I \subset R_n = R$

$$I \leftarrow \bigoplus_j R(-j)^{\beta_{0j}} \leftarrow \bigoplus_j R(-j)^{\beta_{1j}} \leftarrow \cdots$$

Ex If $I=(x_1^5x_2, x_1x_2^5, x_1^2x_2^2)$, then its minimal free res. is

$$I \leftarrow R(-6)\oplus R(-6)\oplus R(-4) \leftarrow R(-7)\oplus R(-7) \leftarrow 0$$

$\beta_{06}=2$, $\beta_{04}=1$, $\beta_{17}=2$

Main target ② Betti numbers・Betti tables

Betti table of I = the table whose (i,j)th entry is $\beta_{i,i+j}(I)$

$\mathrm{reg}(I) = \max\{ j \mid \beta_{i,i+j}(I) \neq 0 \ \exists i\}$ (Castelnuovo-Mumford) regularity

Ex

	i
j	$\cdots$ $\beta_{i,i+j}$

Betti table

$I=(x_1^5x_2, x_1x_2^5, x_1^2x_2^2)$

$\beta_{04}=1$

$\beta_{06}=2$

$\beta_{17}=2$

	0	1	2
4	1	•	•
5	•	•	•
6	2	2	•

Main Question

Q Fix monomials $u_1, u_2, \dots, u_t \in k[x_1,\dots,x_m]$. Let

$\mathfrak{S}_n \cdot u_k = \{\sigma(u_k) \mid \sigma \in \mathfrak{S}_n\}$

$I_n = (\mathfrak{S}_n \cdot u_1) + (\mathfrak{S}_n \cdot u_2) + \cdots + (\mathfrak{S}_n \cdot u_t) \subset R_n = k[x_1,\dots,x_n]$

How the Betti numbers of I_n chang when n increases?

Ex $I_n = (\mathfrak{S}_n \cdot x_1^5x_2) + (\mathfrak{S}_n \cdot x_1^2x_2^2)$

$I_2 = (x_1^5x_2, x_1x_2^5, x_1^2x_2^2)$

$I_3 = (x_1^5x_1, x_1x_2^5, x_1^5x_3, x_1x_3^5, x_2^5x_3, x_2x_3^5, x_1^2x_2^2, x_1^2x_3^2, x_2^2x_3^2)$

$\vdots$

$\beta_{04}(I_n) = \binom{n}{2}$, $\beta_{06}(I_n) = n \times (n-1)$

Ex. Betti tables of $I_n = (\mathfrak{S}_n \cdot x_1^5x_2) + (\mathfrak{S}_n \cdot x_1^2x_2^2)$

n=2

	0	1
4	1	-
5	-	-
6	2	2

$\rightarrow$

n=3

	0	1	2
4	3	-	-
5	-	2	-
6	6	9	-
7	-	-	3

$\rightarrow$

n=4

	0	1	2	3
4	6	-	-	-
5	-	8	-	-
6	12	24	7	-
7	-	-	12	-
8	-	-	-	4

$\rightarrow$

n=5

	0	1	2	3	4
4	10	-	-	-	-
5	-	20	-	-	-
6	20	50	35	5	-
7	-	-	30	4	-
8	-	-	-	20	-
9	-	-	-	-	5

Q Shape of Betti table of I_n for $n \gg 0$?

Some nice things are happening?

n=2

	0	1
4	1	-
5	-	-
6	2	2

$\rightarrow$

n=3

	0	1	2
4	3	-	-
5	-	2	-
6	6	9	-
7	-	-	3

$\rightarrow$

n=4

	0	1	2	3
4	6	-	-	-
5	-	8	-	-
6	12	24	7	-
7	-	-	12	-
8	-	-	-	4

$\rightarrow$

n=5

	0	1	2	3	4
4	10	-	-	-	-
5	-	20	-	-	-
6	20	50	35	5	-
7	-	-	30	4	-
8	-	-	-	20	-
9	-	-	-	-	5

$\rightarrow$

n=6

	0	1	2	3	4	5
4	15	-	-	-	-	-
5	-	40	-	-	-	-
6	30	90	105	30	6	-
7	-	-	60	24	-	-
8	-	-	-	60	5	-
9	-	-	-	-	30	-
10	-	-	-	-	-	6

$\rightarrow$

n=7

	0	1	2	3	4	5	6
4	21	-	-	-	-	-	-
5	-	70	-	-	-	-	-
6	42	149	245	105	42	7	-
7	-	-	105	112	-	-	-
8	-	-	-	140	35	-	-
9	-	-	-	-	105	6	-
10	-	-	-	-	-	42	-
11	-	-	-	-	-	-	7

$\rightarrow$

n=8

	0	1	2	3	4	5	6	7
4	28	-	-	-	-	-	-	-
5	-	112	-	-	-	-	-	-
6	56	224	520	280	168	56	8	-
7	-	-	168	224	-	-	-	-
8	-	-	-	280	140	-	-	-
9	-	-	-	-	280	48	-	-
10	-	-	-	-	-	168	7	-
11	-	-	-	-	-	-	56	-
12	-	-	-	-	-	-	-	8

$\rightarrow$

n=9

	0	1	2	3	4	5	6	7	8
4	36	-	-	-	-	-	-	-	-
5	-	168	-	-	-	-	-	-	-
6	72	324	882	630	504	252	72	9	-
7	-	-	252	504	-	-	-	-	-
8	-	-	-	504	420	-	-	-	-
9	-	-	-	-	630	216	-	-	-
10	-	-	-	-	-	504	63	-	-
11	-	-	-	-	-	-	252	8	-
12	-	-	-	-	-	-	-	72	-
13	-	-	-	-	-	-	-	-	9

Our answer: table nicely decomposes (explained in second part)

n=6

	0	1	2	3	4	5
4	15	-	-	-	-	-
5	-	40	-	-	-	-
6	30	90	105	30	6	-
7	-	-	60	24	-	-
8	-	-	-	60	5	-
9	-	-	-	-	30	-
10	-	-	-	-	-	6

=

	0	1	2	3	4	5
4	15					
5		40				
6			45			
7				24		
8					5	
9						
10						

+

	0	1	2	3	4	5
4						
5						
6	30	60	60	30	6	
7						
8						
9						
10						

+

	0	1	2	3	4	5
4						
5						
6		30				
7			60			
8				60		
9					30	
10						6

$\Downarrow$

n=7

	0	1	2	3	4	5	6
4	21	-	-	-	-	-	-
5	-	70	-	-	-	-	-
6	42	149	245	105	42	7	-
7	-	-	105	112	-	-	-
8	-	-	-	140	35	-	-
9	-	-	-	-	105	6	-
10	-	-	-	-	-	42	-
11	-	-	-	-	-	-	7

=

	0	1	2	3	4	5	6
4	21						
5		70					
6			105				
7				112			
8					35		
9						6	
10							
11							

+

	0	1	2	3	4	5	6
4							
5							
6	42	105	140	105	42	7	
7							
8							
9							
10							
11							

+

	0	1	2	3	4	5	6
4							
5							
6		42					
7			105				
8				140			
9					105		
10						42	
11							7

Motivation ① Noetherian up to symmetry

$R_\infty = k[x_1, x_2, \cdots] \curvearrowleft \mathfrak{S}_\infty$ polynomial ring (with infinitely many variables)

Thm (Cohen 1987, Aschenbrenner-Hillar 2007)
If $I \subset R_\infty$ is $\mathfrak{S}_\infty$-invariant ideal, then I is finitely generated up to symmetry, ie., $\exists f_1, \cdots, f_t$: polynomials s.t.
$$I = (\mathfrak{S}_\infty \cdot f_1) + (\mathfrak{S}_\infty \cdot f_2) + \cdots + (\mathfrak{S}_\infty \cdot f_t)$$

Q algebraic properties of ~~I?~~
$$I_n = (\mathfrak{S}_n f_1) + \cdots + (\mathfrak{S}_n f_t) \quad \text{for } n \gg 0?$$

Motivation ① Noetherian up to symmetry

Q algebraic properties of $I_n = (\mathfrak{S}_n f_1) + \cdots + (\mathfrak{S}_n f_t)$?

Conj (Le-Nagel-Nguyen-Römer 2020)
$f_1, \cdots, f_t$: homogeneous polynomials
$I_n = (\mathfrak{S}_n f_1) + \cdots + (\mathfrak{S}_n f_t)$
Then $\operatorname{reg}(I_n)$ is a linear function on n for $n \gg 0$

Rmk
The conjecture actually consider $k[x_{ij} \mid 1 \le i \le c, j \in \mathbb{N}]$

The question
Fix monomials $u_1, \cdots, u_t$.
How the Betti numbers of $I_n = (\mathfrak{S}_n u_1) + \cdots + (\mathfrak{S}_n u_t)$ change when n increases?

is a special case of this type of questions.

Motivation ② Combinatorial commutative algebra of $\mathfrak{S}_n$-inv. monomial ideals

$\mathfrak{S}_n$-invariant monomial ideals have nice algebraic & combinatorial properties!

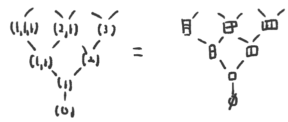
Motivation ② Combinatorial commutative algebra of $\mathfrak{S}_n$-inv. monomial ideals

- $x^{(a_1, a_2, \cdots, a_n)} = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$
- $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{Z}^n$ is a partition $\Leftrightarrow$ $\lambda_1 \ge \cdots \ge \lambda_n \ge 0$

Rem (I will identify a partition with its Young diagram and sometimes omit "0")
$$(3,2,2,0) = (3,2,2) =$$

- $P_n = \{ (\lambda_1, \cdots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \ge \cdots \ge \lambda_n \ge 0 \}$ poset of partitions

The poset P_3



Motivation ② Combinatorial commutative algebra of $\mathfrak{S}_n$-inv. monomial ideals

For $\lambda^1, \cdots, \lambda^t \in P_n$, let
$$\langle \lambda^1, \cdots, \lambda^t \rangle_{\mathfrak{S}_n} = (\mathfrak{S}_n \cdot x^{\lambda^1}) + \cdots + (\mathfrak{S}_n \cdot x^{\lambda^t}) \subset R_n$$

ex $\langle (5,1), (2,2) \rangle_{\mathfrak{S}_3} = (\mathfrak{S}_3 \cdot x^{(5,1)}) + (\mathfrak{S}_3 \cdot x^{(2,2)}) = (x_1^5 x_2, x_1 x_2^5, x_1^2 x_2^2)$

Obs Every $\mathfrak{S}_n$-inv. monomial ideal is generated by partitions

Why? For any monomial x^a, the set $\mathfrak{S}_n x^a$ has (the unique) partition monomial

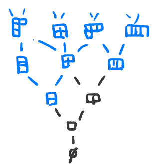
ex $(\underbrace{x_1^5 x_2, x_1 x_2^5, x_1^5 x_3, x_1 x_3^5, x_2^5 x_3, x_2 x_3^5}_{\mathfrak{S}_3\text{-orbit of } x^{(5,1,0)}}, \underbrace{x_1^2 x_2^2, x_1^2 x_3^2, x_2^2 x_3^2}_{\mathfrak{S}_3\text{-orbit of } x^{(2,2,0)}}) = \langle (5,1), (2,2) \rangle_{\mathfrak{S}_3}$

Motivation ② Combinatorial commutative algebra of $\mathfrak{S}_n$-inv. monomial ideals

Obs 2
$$\left\{ \begin{array}{c} \mathfrak{S}_n\text{-inv. monomial} \\ \text{ideals in } R_n \end{array} \right\} \xleftrightarrow{\text{1-to-1}} \left\{ \begin{array}{c} \text{filters of} \\ \text{the poset } P_n \end{array} \right\}$$
$$I \longmapsto \{ \lambda \in P_n \mid x^\lambda \in I \}$$

$F \subset P_n$ is a filter $\Leftrightarrow$ $a \in F$, $a \le b \in P_n \Rightarrow b \in F$

$I = (\mathfrak{S}_3 \cdot x_1^3) + (\mathfrak{S}_3 \cdot x_1 x_2) \longleftrightarrow$
$= \langle$ ▭▭▭ , ⊟ $\rangle_{\mathfrak{S}_3}$



Properties of $\mathfrak{S}_n$-inv. monomial ideals often have a simple description

Ex $I = \langle \lambda^1, \cdots, \lambda^t \rangle_{\mathfrak{S}_n}$, $\ell(\lambda^k) = \#$ of non-zero entries of λ^k
$$\text{Krull-dim}(R_n/I) = \min\{\ell(\lambda^1), \cdots, \ell(\lambda^t)\} - 1$$

Why? $\sqrt{I}$ is the $\mathfrak{S}_n$-inv. monomial ideal $\langle (\underbrace{1, \cdots, 1}_{\ell}, 0, \cdots, 0) \rangle$

ex $I = (\mathfrak{S}_3 \cdot x_1^3 x_2^2) + (\mathfrak{S}_3 \cdot x_1^2 x_2 x_3) \Rightarrow \sqrt{I} = (\mathfrak{S}_3 \cdot x_1 x_2) + (\mathfrak{S}_3 \cdot x_1 x_2 x_3) = \langle (1,1,0) \rangle_{\mathfrak{S}_3}$

More non-trivial result (by Raicu 2020)
Combinatorial formula of regularity and proj. dim. of an $\mathfrak{S}_n$-invariant monomial ideal (does not depend on char(k)).