The problem

> **Q** Fix monomials $u_1, u_2, \cdots, u_t$ and let
> $I_n = (\mathfrak{S}_n \cdot u_1) + \cdots + (\mathfrak{S}_n \cdot u_t)$
> How Betti numbers of I_n changes when n increases?

Ex $\left(I_n = (\mathfrak{S}_n \cdot x_1^5 x_2) + (\mathfrak{S}_n \cdot x_1^2 x_2^2) \right)$

n=2

2	0	1
4	1	-
5	-	-
6	2	2

→ n=3

3	0	1	2
4	3	-	-
5	-	2	-
6	6	9	-
7	-	-	3

→ n=4

4	0	1	2	3
4	6	-	-	-
5	-	8	-	-
6	12	24	7	-
7	-	-	12	-
8	-	-	-	4

→ n=5

5	0	1	2	3	4
4	10	-	-	-	-
5	-	20	-	-	-
6	20	50	35	5	-
7	-	-	30	4	-
8	-	-	-	20	-
9	-	-	-	-	5

→ n=6

6	0	1	2	3	4	5
4	15	-	-	-	-	-
5	-	40	-	-	-	-
6	30	90	105	30	6	-
7	-	-	60	24	-	-
8	-	-	-	60	5	-
9	-	-	-	-	30	-
10	-	-	-	-	-	6

→ n=7

	0	1	2	3	4	5	6
4	21	-	-	-	-	-	-
5	-	70	-	-	-	-	-
6	42	149	245	105	42	7	-
7	-	-	105	112	-	-	-
8	-	-	-	140	35	-	-
9	-	-	-	-	105	6	-
10	-	-	-	-	-	42	-
11	-	-	-	-	-	-	7

→ n=8

	0	1	2	3	4	5	6	7
4	28	-	-	-	-	-	-	-
5	-	112	-	-	-	-	-	-
6	56	224	520	280	168	56	8	-
7	-	-	168	224	-	-	-	-
8	-	-	-	280	140	-	-	-
9	-	-	-	-	280	48	-	-
10	-	-	-	-	-	168	7	-
11	-	-	-	-	-	-	56	-
12	-	-	-	-	-	-	-	8

→ n=9

9	0	1	2	3	4	5	6	7	8
4	36								
5		168							
6	72	324	882	630	504	252	72	9	
7			252	504					
8				504	420				
9					630	216			
10						504	63		
11							252	8	
12								72	
13									9

Our answer: table nicely decomposes

n=6

I_6	0	1	2	3	4	5
4	15	-	-	-	-	-
5	-	40	-	-	-	-
6	30	90	105	30	6	-
7	-	-	60	24	-	-
8	-	-	-	60	5	-
9	-	-	-	-	30	-
10	-	-	-	-	-	6

=

	0	1	2	3	4	5
4	15					
5		40				
6			45			
7				24		
8					5	
9						
10						

+

	0	1	2	3	4	5
4						
5						
6	30	60	60	30	6	
7						
8						
9						
10						

+

	0	1	2	3	4	5
4						
5						
6		30				
7			60			
8				60		
9					30	
10						6

⇓

n=7

	0	1	2	3	4	5	6
4	21	-	-	-	-	-	-
5	-	70	-	-	-	-	-
6	42	149	245	105	42	7	-
7	-	-	105	112	-	-	-
8	-	-	-	140	35	-	-
9	-	-	-	-	105	6	-
10	-	-	-	-	-	42	-
11	-	-	-	-	-	-	7

=

	0	1	2	3	4	5	6
4	21						
5		70					
6			105				
7				112			
8					35		
9						6	
10							
11							

+

	0	1	2	3	4	5	6
4							
5							
6	42	105	140	105	42	7	
7							
8							
9							
10							
11							

+

	0	1	2	3	4	5	6
4							
5							
6		42					
7			105				
8				140			
9					105		
10						42	
11							7

$\mathbb{Z}^n$-gradings

Consider the $\mathbb{Z}^n$-grading of R_n given by $\deg(x_i) = e_i$

For a monomial ideal $I \subset R_n$ and $a = (a_1, \cdots, a_n) \in \mathbb{Z}^n$, let

$$\beta_{i,a}(I) = \dim_k \operatorname{Tor}_i(I, k)_a \quad \text{$\mathbb{Z}^n$-graded Betti number}$$

> **Rem** $\beta_{ij} = \sum_{|a|=j} \beta_{i,a}$

Obs When I is $\mathfrak{S}_n$-inv., to know all $\mathbb{Z}^n$-graded Betti numbers, it is enough to know $\beta_{i(a_1,\cdots,a_n)}(I)$ with $a_1 \geq a_2 \geq \cdots \geq a_n$

Why $\mathfrak{S}_n$-action to $\operatorname{Tor}_i(I,k)$ permutes $\mathbb{Z}^n$ gradings, e.g.,

$$\operatorname{Tor}_i(I,k)_{(2,2,1)} \cong \operatorname{Tor}_i(I,k)_{(2,1,2)} \cong \operatorname{Tor}_i(I,k)_{(1,2,2)}$$

Ex. Non-zero positions of multigraded Betti table of (up to permutations)

Betti table

I_2	0	1
4	1	-
5	-	-
6	2	2

→

I_3	0	1	2
4	3	-	-
5	-	2	-
6	6	9	-
7	-	-	3

→

I_4	0	1	2	3
4	6	-	-	-
5	-	8	-	-
6	12	24	7	-
7	-	-	12	-
8	-	-	-	4

> **Rem** $\beta_{i(a_1,\cdots,a_n)}(I_n) = \beta_{i(a_1,\cdots,a_n,0)}(I_{n+1})$

multidegrees

- I_2: (2,2); (5,1); (5,2)
- I_3: (2,2,0) (2,2,2); (5,1,0) (5,1,1); (5,2,0) (5,2,2)
- I_4: (2,2,0,0), (2,2,2,0), (2,2,2,2); (5,1,0,0), (5,1,1,0), (5,1,1,1); (5,2,0,0), (5,2,2,0), (5,2,2,2)

> **Q** $\beta_{i,(a_1,\cdots,a_n)}(I_n) \neq 0 \Leftrightarrow \beta_{i+1,(a_1,\cdots,a_n,a_n)}(I_{n+1}) \neq 0$?

First Result

← determine the shape of $\mathbb{Z}^n$-graded Betti table of I_{n+1} from the table of I_n

> **Thm** (M, 2020) $u_1, \cdots, u_t$: monomials in $k[x_1, \cdots, x_m]$
> $I_n = (\mathfrak{S}_n \cdot u_1) + \cdots + (\mathfrak{S}_n \cdot u_t)$ $(n \geq m)$
> $a_1 \geq a_2 \geq \cdots \geq a_n$.
> (1) $\beta_{i(a_1,\cdots,a_n)}(I_n) \neq 0 \Rightarrow \beta_{i(a_1,\cdots,a_n,0)}(I_{n+1}) \neq 0$ ← trivial
> (2) if $a_n > 0$, then
> $\beta_{i(a_1,\cdots,a_n)}(I_n) \neq 0 \Rightarrow \beta_{i+1,(a_1,\cdots,a_n,a_n)}(I_{n+1}) \neq 0$
> (3) if $0 < a_{n+1} < a_n$ then
> $\beta_{i(a_1,\cdots,a_n,a_{n+1})}(I_{n+1}) = 0$ for all i

Ex, $I_n = (\mathfrak{S}_n \cdot x_1^5 x_2) + (\mathfrak{S}_n \cdot x_1^2 x_2^2)$

Betti table

I_2	0	1
4	1	-
5	-	-
6	2	2

I_3	0	1	2
4	3	-	-
5	-	2	-
6	6	9	-
7	-	-	3

multidegrees

- (2,2); (5,1); (5,2)
- (2,2,0) (2,2,2); (5,1,0) (5,1,1); (5,2,0) (5,2,2)

↑ by (1) ↑ by (2)

> **Thm**
> (1) $\beta_{i(a_1,\cdots,a_n)}(I_n) \neq 0 \Rightarrow \beta_{i(a_1,\cdots,a_n,0)}(I_{n+1}) \neq 0$
> (2) if $a_n > 0$, then
> $\beta_{i(a_1,\cdots,a_n)}(I_n) \neq 0 \Rightarrow \beta_{i+1,(a_1,\cdots,a_n,a_n)}(I_{n+1}) \neq 0$
> (3) if $0 < a_{n+1} < a_n$ then
> $\beta_{i(a_1,\cdots,a_n,a_{n+1})}(I_{n+1}) = 0$ for all i

Ex, $I_6 = (\mathfrak{S}_6 \cdot x_1^5 x_2) + (\mathfrak{S}_6 \cdot x_1^2 x_2^2)$

I_6	0	1	2	3	4	5
4	15	-	-	-	-	-
5	-	40	-	-	-	-
6	30	90	105	30	6	-
7	-	-	60	24	-	-
8	-	-	-	60	5	-
9	-	-	-	-	30	-
10	-	-	-	-	-	6

=

	0	1	2	3	4	5
4	①					
5		②				
6			③			
7				④		
8					⑤	
9						
10						

+

	0	1	2	3	4	5
4						
5						
6	①	②	③	④	⑤	
7						
8						
9						
10						

+

	0	1	2	3	4	5
4						
5						
6		①				
7			②			
8				③		
9					④	
10						⑤

I_2	0	1
4	1	-
5	-	-
6	2	2

(2,2)
(5,1) (5,2)

① (2,2,0,0,0,0)
② (2,2,2,0,0,0)
③ (2,2,2,2,0,0)
④ (2,2,2,2,2,0)
⑤ (2,2,2,2,2,2)

① (5,1,0,0,0,0)
② (5,1,1,0,0,0)
③ (5,1,1,1,0,0)
④ (5,1,1,1,1,0)
⑤ (5,1,1,1,1,1)

① (5,2,0,0,0,0)
② (5,2,2,0,0,0)
③ (5,2,2,2,0,0)
④ (5,2,2,2,2,0)
⑤ (5,2,2,2,2,2)

Ex, $I_6 = (\mathfrak{S}_6 \cdot x_1^5 x_2) + (\mathfrak{S}_6 \cdot x_1^2 x_2^2)$

I_6	0	1	2	3	4	5
4	15	-	-	-	-	-
5	-	40	-	-	-	-
6	30	90	105	30	6	-
7	-	-	60	24	-	-
8	-	-	-	60	5	-
9	-	-	-	-	30	-
10	-	-	-	-	-	6

=

	0	1	2	3	4	5
4	15					
5		40				
6			45			
7				24		
8					5	
9						
10						

+

	0	1	2	3	4	5
4						
5						
6	30	60	60	30	6	
7						
8						
9						
10						

+

	0	1	2	3	4	5
4						
5						
6		30				
7			60			
8				60		
9					30	
10						6

I_2	0	1
4	1	-
5	-	-
6	2	2

(2,2)
(5,1) (5,2)

① (2,2,0,0,0,0)
② (2,2,2,0,0,0)
③ (2,2,2,2,0,0)
④ (2,2,2,2,2,0)
⑤ (2,2,2,2,2,2)

① (5,1,0,0,0,0)
② (5,1,1,0,0,0)
③ (5,1,1,1,0,0)
④ (5,1,1,1,1,0)
⑤ (5,1,1,1,1,1)

① (5,2,0,0,0,0)
② (5,2,2,0,0,0)
③ (5,2,2,2,0,0)
④ (5,2,2,2,2,0)
⑤ (5,2,2,2,2,2)

How to compute numbers?

Our answer : Numbers can be computed by looking representations

I_6	0	1	2	3	4	5
4	15	-	-	-	-	-
5	-	40	-	-	-	-
6	30	90	105	30	6	-
7	-	-	60	24	-	-
8	-	-	-	60	5	-
9	-	-	-	-	30	-
10	-	-	-	-	-	6

=

	0	1	2	3	4	5
4	15					
5		40				
6			45			
7				24		
8					5	
9						
10						

① (2,2,0,0,0,0)
② (2,2,2,0,0,0)
③ (2,2,2,2,0,0)
④ (2,2,2,2,2,0)
⑤ (2,2,2,2,2,2)

+

	0	1	2	3	4	5
4						
5						
6	30	60	60	30	6	
7						
8						
9						
10						

① (5,1,0,0,0,0)
② (5,1,1,0,0,0)
③ (5,1,1,1,0,0)
④ (5,1,1,1,1,0)
⑤ (5,1,1,1,1,1)

+

	0	1	2	3	4	5
4						
5						
6		30				
7			60			
8				60		
9					30	
10						6

① (5,2,0,0,0,0)
② (5,2,2,0,0,0)
③ (5,2,2,2,0,0)
④ (5,2,2,2,2,0)
⑤ (5,2,2,2,2,2)

Idea to understand numbers : look $\mathfrak{S}_n$-module structure

■ S^λ = Specht module w.r.t. a partition λ (irreducible $\mathfrak{S}_{|\lambda|}$-module)

■ $S^{(\lambda^1,\dots,\lambda^t)} = S^{\lambda^1}\otimes\cdots\otimes S^{\lambda^t}$ (irr. $(\mathfrak{S}_{|\lambda^1|}\times\cdots\times\mathfrak{S}_{|\lambda^t|})$-mod)

■ A hook partition is a partition of the form $\lambda=(a,1,1,\dots,1)$

■ H_n = the set of hooks with $|\lambda|=n$

$H_4 = \{$  $\}$

Idea to understand numbers : look $\mathfrak{S}_n$-module structure

I : $\mathfrak{S}_n$-invariant monomial ideal. Assume $\mathrm{char}(k)=0$.

Obs (1) $\mathrm{Tor}_i(I,k)$ is an $\mathfrak{S}_n$-module

(2) To know all $\mathbb{Z}^n$-graded Betti numbers it is enough to know $\mathrm{Tor}_i(I,k)_{(a_1,\dots,a_n)}$ with $a_1\ge\cdots\ge a_n$

(3) Set $\mathbf{a}=(d_1^{r_1},d_2^{r_2},\dots,d_m^{r_m}) = (\underbrace{d_1,\dots,d_1}_{r_1},\underbrace{d_2,\dots,d_2}_{r_2},\dots,\underbrace{d_m,\dots,d_m}_{r_m})$

Then $\mathrm{Tor}_i(I,k)_{\mathbf{a}}$ is an $(\mathfrak{S}_{r_1}\times\cdots\times\mathfrak{S}_{r_m})$-module.

$\Rightarrow$ $\mathrm{Tor}_i(I,k)_{\mathbf{a}}$ decomposes into irreducible $(\mathfrak{S}_{r_1}\times\cdots\times\mathfrak{S}_{r_m})$-modules

product of irr. $\mathfrak{S}_{r_i}$-mod

Representation of $\mathrm{Tor}_i(I,k)_{\mathbf{a}}$

S^λ: Specht module

H_r: set of hooks of size r

Thm (M-Raicu) Assume $\mathrm{char}(k)=0$.
I: $\mathfrak{S}_n$-inv. monomial ideal,
$\mathbf{a}=(d_1^{r_1},d_2^{r_2},\dots,d_m^{r_m})$ $(d_1>\cdots>d_m)$.
We have a decomposition of $(\mathfrak{S}_{r_1}\times\cdots\times\mathfrak{S}_{r_m})$-modules

$$\mathrm{Tor}_i(I,k)_{\mathbf{a}} \simeq \bigoplus_{\pi=(h_1,\dots,h_m)\in H_{r_1}\times\cdots\times H_{r_m}} \left(S^{h_1}\otimes\cdots\otimes S^{h_m}\right)^{\gamma_i^{\pi,\mathbf{a}}}$$

Moreover $\gamma_i^{\pi,\mathbf{a}}$ is equal to the k-dim of a reduced homology group of some simplicial complex $\Delta^{\pi,\mathbf{a}}(I)$

※ I omit the definition of the simplicial complex $\Delta^{\pi,\mathbf{a}}(I)$.

Example $I_2 = (\mathfrak{S}_2\cdot x_1^5x_2)+(\mathfrak{S}_2\cdot x_1^2x_2^2)$

	0	1
4	1	-
5	-	-
6	2	2

(2,2)
(5,1) (5,2)

$\mathrm{Tor}_0(I,k)_{(2,2)} \simeq S^{\square\square}$ (trivial $\mathfrak{S}_2$-mod)

$\mathrm{Tor}_0(I,k)_{(5,1)} \simeq S^{\square}\otimes S^{\square}$

$\mathrm{Tor}_1(I,k)_{(5,2)} \simeq S^{\square}\otimes S^{\square}$

Example : $I_n = (\mathfrak{S}_n\cdot x_1^5x_2)+(\mathfrak{S}_n\cdot x_1^2x_2^2)$

$(\boxplus,\square) = S^{\boxplus}\otimes S^{\square}$

Betti table

I_2	0	1
4	1	-
5	-	-
6	2	2

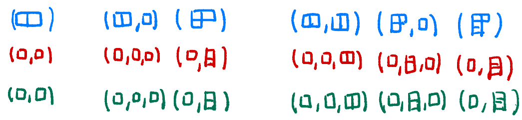
I_3	0	1	2
4	3	-	-
5	-	2	-
6	6	9	-
7	-	-	3

I_4	0	1	2	3
4	6	-	-	-
5	-	8	-	-
6	12	24	7	-
7	-	-	12	-
8	-	-	-	4

Non-zero multidegrees

I_2	I_3	I_4
(2,2)	(2,2,0) (2,2,2)	(2,2,0,0), (2,2,2,0), (2,2,2,2)
(5,1)	(5,1,0) (5,1,1)	(5,1,0,0), (5,1,1,0), (5,1,1,1)
(5,2)	(5,2,0) (5,2,2)	(5,2,0,0), (5,2,2,0), (5,2,2,2)

Representation



Final Result

Thm (M-Raicu) $u_1,\dots,u_t$: monomials

$I_n=(\mathfrak{S}_n\cdot u_1)+\cdots+(\mathfrak{S}_n\cdot u_t)\subset R_n$

$\mathbf{a}=(a_1,\dots,a_n,a_{n+1})=(d_1^{r_1},d_2^{r_2},\dots,d_m^{r_m})$

$\pi=(h_1,\dots,h_m=(p,1^q))\in H_{r_1}\times H_{r_2}\times\cdots\times H_{r_m}$

(1) if $a_{n+1}=0$ and $h_m=(r_m)$, then
$\gamma_i^{\pi,\mathbf{a}}(I_{n+1})=\gamma_i^{(h_1,\dots,h_{m-1},(r_m-1)),(a_1,\dots,a_n)}(I_n)$

(2) if $a_{n+1}=a_n$ and $h_m=(p,1^q)$ with $q>0$ then
$\gamma_i^{\pi,\mathbf{a}}(I_{n+1})=\gamma_{i-1}^{(h_1,\dots,h_{m-1},(p,1^{q-1})),(a_1,\dots,a_n)}(I_n)$

(3) $\gamma_i^{\pi,\mathbf{a}}(I_{n+1})=0$ for other cases.

Rem $\mathrm{Tor}_i(I,k)_{\mathbf{a}} \simeq \bigoplus_{\pi\in H_{r_1}\times\cdots\times H_{r_m}} (S^\pi)^{\gamma_i^{\pi,\mathbf{a}}}$

Set $\gamma_i^{\pi,\mathbf{a}}=\gamma_i^{\pi,\mathbf{a}}(I)$

Example : $I_n = (\mathfrak{S}_n\cdot x_1^5x_2)+(\mathfrak{S}_n\cdot x_1^2x_2^2)$

(1) if $a_{n+1}=0$ and $h_m=(r_m)$, then $\gamma_i^{\pi,\mathbf{a}}(I_{n+1})=\gamma_i^{(h_1,\dots,h_{m-1},(r_m-1)),(a_1,\dots,a_n)}(I_n)$
(2) if $a_{n+1}=a_n$ and $h_m=(p,1^q)$ with $q>0$ then $\gamma_i^{\pi,\mathbf{a}}(I_{n+1})=\gamma_{i-1}^{(h_1,\dots,h_{m-1},(p,1^{q-1})),(a_1,\dots,a_n)}(I_n)$
(3) $\gamma_i^{\pi,\mathbf{a}}(I_{n+1})=0$ for other cases

Betti table

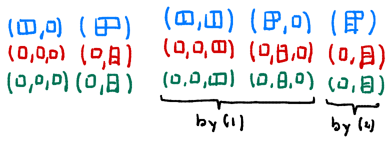
I_3	0	1	2
4	3	-	-
5	-	2	-
6	6	9	-
7	-	-	3

I_4	0	1	2	3
4	6	-	-	-
5	-	8	-	-
6	12	24	7	-
7	-	-	12	-
8	-	-	-	4

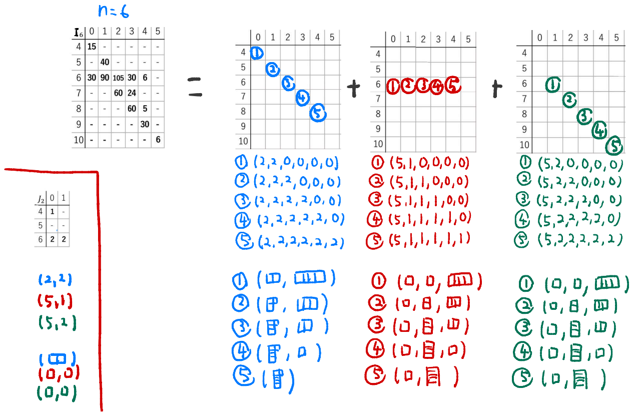
multidegrees

I_3	I_4
(2,2,0) (2,2,2)	(2,2,0,0), (2,2,2,0), (2,2,2,2)
(5,1,0) (5,1,1)	(5,1,0,0), (5,1,1,0), (5,1,1,1)
(5,2,0) (5,2,2)	(5,2,0,0), (5,2,2,0), (5,2,2,2)

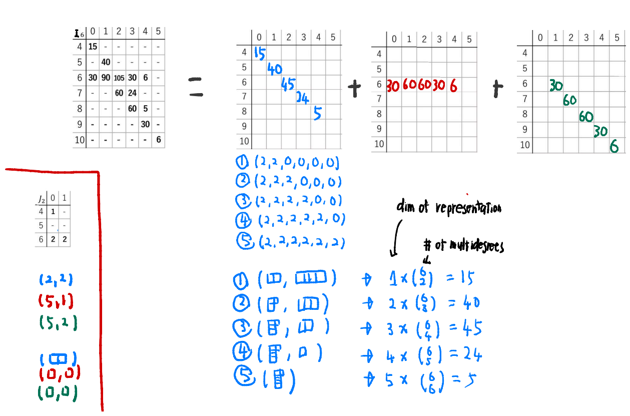
Representation



Example : $I_n = (\mathfrak{S}_n \cdot x_1^5 x_2) + (\mathfrak{S}_n \cdot x_1^2 x_2^2)$



Example : $I_n = (\mathfrak{S}_n \cdot x_1^5 x_2) + (\mathfrak{S}_n \cdot x_1^2 x_2^2)$



Idea of Proof

(2) if $a_{n+1} = a_n$ and $h_m = (p, 1^q)$ with $q > 0$ then
$$\gamma_i^{\pi,a}(I_{n+1}) = \gamma_{i-1}^{(h_1,\cdots,h_{m-1},(p,1^{q-1})),(a_1,\cdots,a_n)}(I_n)$$

Thm

(1) if $a_{n+1} = 0$ and $h_m = (r_m)$, then
$$\gamma_i^{\pi,a}(I_{n+1}) = \gamma_i^{(h_1,\cdots,h_{m-1},(r_m-1)),(a_1,\cdots,a_n)}(I_n)$$
(2) if $a_{n+1} = a_n$ and $h_m = (p, 1^q)$ with $q > 0$ then
$$\gamma_i^{\pi,a}(I_{n+1}) = \gamma_{i-1}^{(h_1,\cdots,h_{m-1},(p,1^{q-1})),(a_1,\cdots,a_n)}(I_n)$$
(3) $\gamma_i^{\pi,a}(I_{n+1}) = 0$ for other cases

Set $\pi' = (h_1,\cdots,h_{m-1},(p,1^{q-1}))$ and $a' = (a_1,\cdots,a_n)$.

$\gamma_{i+1}^{\pi,a}(I_{n+1})$ = dim of homology of some simplicial complex $\Delta^{\pi,a}(I_{n+1})$

$\gamma_i^{\pi',a'}(I_n)$ = dim of homology of some simplicial complex $\Delta^{\pi',a'}(I_n)$

Prove
$$\Delta^{\pi,a}(I_{n+1}) = \Delta^{\pi',a'}(I_n)$$

Problem

- Similar result for non-monomial case ?
- Any relation between $\mathrm{Tor}_i(I_n, k)$ and $\mathrm{Tor}_{i+1}(I_{n+1}, k)$?

Thank you for your attension !