$\boxed{\text{Global } +\text{-regularity}}$

j/w Bhatt, Ma, Patakfalvi, Schwede, Waldron, Witaszek

Local Motivations:

R ring w/ char $p > 0$

$F : R \longrightarrow R$ Frobenius (algebra map)

$r \longmapsto r^p$

$\updownarrow$ restr of scalars ~~for~~ F

$R \longrightarrow F_* R$ (R-mod map)

R is F-split if this admits an R-mod retraction.

<u>stronger</u>: (strong) F-regularity

R is strongly F-regular if for all $c \in R \setminus$ (any min prime)

$$\begin{tikzcd} R \arrow[r, "F^e"] & F^e_* R \arrow[r, "\cdot F^e_* c"] & F^e_* R \\ 1 \arrow[mapsto, rr] & & F^e_* c \end{tikzcd}$$

this splits for $e \gg 0$ (depending on c).

Thm (Hochster-Huneke)

regular $\Rightarrow$ strong F-regular $\Rightarrow$ for any mod finite extn $R \hookrightarrow S$ splits

+ Q-Gor (Singh)

"splinter condition"

Thm (Direct Summand Thm) (Hochster , André)

R regular, any char $\implies$ satisfies splinter condition

2

very characteristic dependent

Splinters in various characteristics:

- $\mathbb{Q} \subseteq R$ splinter $\Longleftrightarrow$ normal
  $\not\Longrightarrow$ CM
- $(R, \mathfrak{m})$ domain local w/ $p \in \mathfrak{m}$

Thm (Bhatt) $K = \operatorname{Frac} R$, $R^+ =$ int clos of R in $\overline{K}$

R^+ is a (big) CM algebra.

Global Definitions:

X = Noeth, normal, F-finite scheme char $p > 0$

F is finite

$\Delta \geq 0$ eff $\mathbb{Q}$-div on X

(X, Δ) is globally F-reg if $\forall$ $D \geq 0$ cartier divisor $\exists e > 0$

$$\mathcal{O}_X \xrightarrow{F^e} F_*^e \mathcal{O}_X \xrightarrow{c} F_*^e \mathcal{O}_X(\lceil (p^e - 1)\Delta \rceil + D)$$

splits.

Example: (1) $X = \operatorname{Spec} R$

$\Delta = 0$

globally F-reg = strongly F-reg

(2) X smooth, proj $/ k = \bar{k}$

char $k = p > 0$

(X, Δ) globally F-reg $\Rightarrow$ (X, Δ) log Fano

$-K_X - \Delta$ ample

e.g. $\dim X = 1$

$\Delta = 0$

X gl F-reg $\Leftrightarrow$ $X = \mathbb{P}^1$

<u>Setup:</u> X Noeth, normal integral scheme proper / (R, m) complete local w/ char $R/m = p > 0$

$\Delta \geq 0$ eff $\mathbb{Q}$-div on X

__Def:__ (X, Δ) globally $+$-reg if for all $f: Y \to X$ finite, Y normal integral, $\mathcal{O}_X \to f_*\mathcal{O}_Y \to f_*\mathcal{O}_Y(\lfloor f^*\Delta \rfloor)$ splits.

__Ex:__ (1) (X, Δ) globally F-reg $\Longrightarrow$ globally $+$-reg, in char $p>0$

② R regular, $X = \operatorname{Spec} R$

$\Longrightarrow$ X globally F-regular

Can generalize to account for direct summands of regular rings, e.g. rational double points of char $p > 5$.

<u>Thm</u>: (X, Δ) glob t-regular; $K_X + \Delta$ $\mathbb{Q}$-Cartier; L line bundle so that $L - (K_X + \Delta)$ big + semiample.

$$\Rightarrow H^i(X, L) = 0 \quad i > 0.$$

+-stable Sections of Line Bundles:

(same setup: X Noeth normal integral proper / $(R,\mathfrak{m})$ complete local w/ char $R/\mathfrak{m} = p > 0$, $\Delta \geq 0$ ($\mathbb{Q}$-div).

M Cartier divisor

$$B^0(X, \Delta, \mathcal{O}_X(M)) = \bigcap_{\substack{f: Y \to X \\ \text{finite surj}}} \operatorname{im}\left(H^0(Y, \mathcal{O}_Y(K_Y - f^*(K_X+\Delta) + f^*M)) \to H^0(X, \mathcal{O}_X(M)) \right)$$

Some properties!

1. (X, Δ) globally $+$-reg
$$\implies B^0(X, \Delta; M) = H^0(X, M)$$
$\forall$ M.

2. If $K_X + D$ $\mathbb{Q}$-Cartier, can also use alterations.

3. B^0 transforms "as expected" for finite maps + alterations.

(4) X proj / $\mathrm{Spec}\,k$ and is regular, $B^0(X, \omega_X \otimes \mathcal{L}^{\otimes n})$

$\parallel$

$H^0(X, \omega_X \otimes \mathcal{L}^n)$

for $\mathcal{L}$ ample, $n \gg 0$.

Example: E elliptic curve / $\mathbb{Z}_p$.

(a) $B^0(E, \mathcal{O}_E) = \mathcal{O}$

(b) $B^0(E_{\mathbb{Q}_p}, \mathcal{O}_{E_{\mathbb{Q}_p}}) = \mathbb{Q}_p$.

More on vanishing:

Say $\mathcal{L}$ ample on X.

$$S = \bigoplus_{i \geq 0} H^0(X, \mathcal{L}^i), \quad \mathfrak{m}_S \text{ homog max ideal}$$

thm (Bhatt, ...) $H^i_{\mathfrak{m}_S}(S^+/p^r) = 0$ $i < \dim S$

and if $i < \dim X$

$$H^i_{\mathfrak{m}_R}(R\Gamma(X^+, \pi^*\mathcal{L}^{-1})) = 0$$

dual to KV on X^+

Consequence: Get lifting thms for sections of B^0 from divisors.

Thm (w/ appropriate assumptions)
If A ample Cartier on X

$$B^0_S(X, S; \mathcal{O}_X(K_X + S + A)) \twoheadrightarrow B^0(S; \mathcal{O}_S(K_S + A|_S))$$

└ "adjoint version of B^0"

Applications!

① Case of Fujita's Conj in mixed char

Thm X regular d-dim'l proj flat / (R, m). $\mathcal{L}$ ample s.s.

$\Rightarrow B^0(X, \omega_X \otimes \mathcal{L}^{d+1})$ glob gens

$\omega_X \otimes \mathcal{L}^{d+1}$.

② Seshadri constants

③ MMP in dim 3 mixed
char $p > 5$.

④ (X, Δ) affine 3-dim'l
$X = \operatorname{Spec} R$
KLT pair,
mixed char $p > 5$.

$\Longrightarrow$ $\forall$ Weil div
D on X, $\bigoplus_{i \geq 0} \mathcal{O}_X(iD)$ fin gen.