On The Size and Shape of Betti Numbers

Notation: $S = k[x_1, \ldots, x_n]$ $\boxed{n = \dim = \#\text{vars.}}$

M f.g. graded S-module. (E.g. $M = S/I$)

Big Question: How big are the betti numbers $\beta_i(M)$?

How big is $\sum \beta_i(M)$?

$$S/I \leftarrow S \longleftarrow S^{\beta_1} \nwarrow K$$

§1. What are Betti #s?

S/I has a minimal free resolution:

n = # vars.

$$0 \leftarrow S/I \leftarrow S^1 \leftarrow S^{\beta_1} \leftarrow S^{\beta_2} \leftarrow \cdots \leftarrow S^{\beta_n} \leftarrow 0$$

$\beta_0 = 1$

$\beta_1 = \#$ min generators of I

$\beta_2 = \#$ syzygies/relations on the generators

- (Hilbert Syzygy Thm): $\beta_i = 0$ if $i > n$.
- β_i and their degrees encode the Hilbert function (+ more) of S/I.

$$S/I \text{ f.g. } S\text{-module} \xrightarrow{\text{Hilb. syz. thm}} \{\beta_0, \beta_1, \ldots, \beta_n\}$$

$\beta_1(S/I) = \#$ min. gens of $I \geq$ ht $I = c = \operatorname{codim}(S/I)$

Krull Altitude Thm.

(Auslander–Buchsbaum) $\operatorname{pdim}(S/I) \geq c$

$\beta_c \geq 1$.

(Koszul Complex) If $\beta_1(S/I) = c$ i.e. I is gen'd by a regular sequence then $\beta_i(S/I) = \binom{c}{i}$

$\beta(S/I)$

General S/I

$\beta_0 \geq 1$

$\beta_1 \geq c$

$\beta_i \geq \binom{c}{i}$

$\beta_c \geq 1$

E.g.

$\beta_0 = 1$

$\beta_1 = c$

$\beta_2 = \binom{c}{2}$

$\beta_i = \binom{c}{i}$

$\beta_c = 1$

$S/I = $ C.I.

$I = (f_1, \ldots, f_c)$

ht $I = c$

(Koszul Complex)

"The Koszul Complex is the smallest resolution"

Conjecture (Buchsbaum-Eisenbud Horrocks Rank Conjecture) (1977)

If $I \subset S$ has height c, then

$$\underline{\beta_i(S/I)} \geq \underline{\binom{c}{i}}$$

Known for $c = 1, 2, 3, 4$, open in general for $c \geq 5$

$c \geq 5$: Special Cases:

- The min resolution of S/I has an algebra structure. (Buchsbaum-Eisenbud, '77)
- I monomial (proof follows by "polarize then localize")
- I is licci (Huneke-Ulrich '87)
- $\operatorname{reg}(S/I)$ is low relative to degrees of gens of I (Erman '08)

* It's open to prove that if $I = (Q_1, \ldots, Q_b)$ in 5 vars then $\beta_2(S/I) \geq \binom{5}{2}$ (Dugger)

A Weaker Conjecture (Total Rank Conjecture)

$$\sum_i \beta_i(S/I) \geqslant 2^c \qquad c = \operatorname{codim}(S/I)$$

- Known for $\underline{c=5}$ (Avramov-Buchweitz)
- 2018 Walker $\forall c$, provided char $k \neq 2$.

If char $k \neq 2$ then $\sum_i \beta_i(M) \geqslant 2^c$ <u>and</u>

equality holds $\iff$ $M \cong S/(f_1, \dots, f_c)$ complete intersection.

$\Updownarrow$ $\beta_i(M) = \binom{c}{i}$

- If S/I is <u>not</u> a CI then

$$\sum \beta_i(S/I) \geqslant 2^c + 2$$

<u>Q</u>: Is this inequality ever achieved if $c \geqslant 3$.

Examples: Some ideals of height $c=3$:

	β_0		β_1		β_2		β_3	$\Sigma \beta_i$	
$S/I \leftarrow$	S^1	$\leftarrow$	S^3	$\leftarrow$	S^3	$\leftarrow$	S^1	$\boxed{8}$	CI 2^3
$S/I \leftarrow$	1	$\leftarrow$	4	$\leftarrow$	5	$\leftarrow$	2	12	
$S/I \leftarrow$	1		5		5		1	12	
$S/I \leftarrow$	1		5		6		2	14	
$S/I \leftarrow$	1		6		8		3	18	

Some Examples of Ideals generated by quadrics

Some ideals of height $c=4$:

		β_0		β_1		β_2		β_3		β_4	$\Sigma \beta_i$	
	$S/I \leftarrow$	S^1	$\leftarrow$	S^4	$\leftarrow$	S^6	$\leftarrow$	S^4	$\leftarrow$	S^1	$\boxed{16}$	2^4
5 gen'd $\rightarrow$		1		5		15		16		5	42	
6 gen'd $\rightarrow$		1		6		10		6		1	$\boxed{24}$	
		1		6		13		12		4	36	
		1		7		11		8		3	30	

§ Monomial Ideals (Multigraded Case)

$I \subseteq k[x_1, \ldots, x_n]$

$c = \text{ht}\, I$, $c \leq n$.

Thm: (Charalambous - Evans '92), I not a CI then if I is monomial and S/I has finite length then

1) $\beta_i(S/I) \geq \binom{c}{i} + \binom{c-1}{i-1} \quad \forall i$

2) $\sum_i \beta_i(S/I) \geq 2^c + 2^{c-1} = 1.5(2^c)$

	I monomial S/I fin. len	S/I finite length	S/I monomial
1)	true	false 1 5 5 1	false 1 5 5 1
2)	true	open	true (B-Seiner) open for multigraded modules.

Question (Charalambous - Evans - Miller)

If M is a module of codim c, and $M \not\cong S/(f_1, \ldots, f_c)$ then is $\sum_i \beta_i(M) \geq 2^c + 2^{c-1}$?

(CEM) True if $c \leq 4$ ($c = 4$ proof uses a classification of Tor algebra structures)

$\sum \beta_i \geq 24$ if $c = 4$

- If I is monomial
  S/I finite length $\Longrightarrow$ S/I is a C.I.
  Gorenstein

(false without both assumptions)

Low Regularity Case

Thm (Erman, 2009)

If M is a module of codim c, of low regularity then

$$\beta_i(M) \geq \beta_0(M)\binom{c}{i}$$

(uses Boij–Söderberg Theory)

Low Regularity Means:

- M is generated in degree 0
- a = least degree of a minimal syzygy. $(a \geq 2)$
- $\operatorname{reg}(M) \leq 2a-2$

0	*	-	-			
1		-	-			
$\vdots$		-	-			
$a-1$		$\boxed{*}$	*		*	*
		*	*	......	*	*
		*	*		*	*
		*	*		*	*
$2a-2$		*	*		*	*

1				
*	*	*	*	
*	*	*	*	

Includes all Veronese Embeddings of $\mathbb{P}^n$, high degree curves, ...

Question (Huneke): Can you relax the regularity bound to prove $\sum \beta_i(M) \geq 2^c$? (No?)

Question: Can one use the low regularity assumption to prove

$$\sum_i \beta_i(M) \geq 2^c + 2^{c-1}$$

Thm: (B–Wigglesworth)

If $\boxed{c \geq 3}$, if M has low regularity then

$$\boxed{\sum_i \beta_i(M) \geq 2^c + 2^{c-1}}$$

$c=4$

① $\beta_i \geq \binom{c}{i} + \binom{c-1}{i-1}$? no

② $\beta_i \geq 1.5\binom{c}{i}$ no but only if $i \geq \lceil \frac{c}{2} \rceil$

③ $\beta_i \geq 2\binom{c}{i}$ for first half

$\geq 1\binom{c}{i}$ for last half (Erman)

on average $\beta_i \geq 1.5\binom{c}{i}$

Thm: if $\boxed{c \geq 9}$ and $1 \leq i \leq \lceil \frac{c}{2} \rceil$, then

$$\beta_i(M) \geq 2\binom{c}{i}$$ (+ low regularity)

Curious Application:

Say $I \subset k[x_1, \dots, x_n]$ and I gen'd in deg 2,

and $m^3 \subseteq I$. (low regularity)

is it possible for I to have $< 2\binom{n}{1}$ generators?

No if $n \geq 9$

yes if $n = 8$ I with 15 generators in 8 vars.

$I \subset k[x_1, \ldots, x_{10}]$ $\qquad I = (Q_1, \ldots, Q_{16})$

$\uparrow$

§ Proof:

Idea: M module of codim c:

$$\beta\text{-table}(M) = \sum_{\underline{d}} \lambda_{\underline{d}}^{\in \mathbb{Q}_{\geq 0}} \underbrace{\left(\begin{array}{c}\beta\text{-tables of} \\ \text{pure modules}(\underline{d})\end{array}\right)}_{\text{these are known explicit Rational functions.}}$$

sufficient to show

$$\underbrace{\frac{\beta_i(\text{pure diagram } \underline{d})}{\binom{c}{i}}}_{\text{Rational function } n+1 \text{ variables.}} \geq 2.$$

first half.

$\vdots$

$$F(c, a, \overset{\beta_i}{i}, b, e) \geq 2$$

$$F(a, b, e, \overset{c}{n}, i) = \frac{(a)\cdots(a+(i-2))}{(b+1)\cdots(b+(i-1))}\,\frac{(n+1)\cdots(n+a+b+e-1)}{(i+1)\cdots(i+a+b+e-1)}\,\frac{e!}{(n-i+1)\cdots((n-i)+e)}.$$

Known for $c = 1, 2, 3, 4$. in general. (easy)

For $c \geq 5$ it is known in some cases:

- The min res. of S/I has an algebra structure.
  Buchsbaum-Eisenbud ('77)
- I is licci Huneke-Ulrich ('87)
- I is monomial (classical)
- $\operatorname{reg}(S/I)$ is "small" (Erman '08)
- Syzygy Thm (Evans-Griffith '81) says $\beta_i \geq 2i+1$
  $(i < c)$

Weaker Conjecture $(c = ht\, I)$

Total Rank Conjecture $\sum \beta_i \geq 2^c$

Now a theorem of Walker (2018)
(provided char $k \neq 2$)

$\to$ If S/I is not a complete intersection (CI) then the inequality is strict.

Stronger Bounds: $\boxed{(\text{If } S/I \text{ is not a CI})}$

Charalambous - Evans ('92) $I \subset k[x_1, \ldots, x_n]$
$c = ht\, I$.

If I is monomial $c = n$ and S/I not a CI
then $\beta_i(S/I) \geq \binom{c}{i} + \binom{c-1}{i-1}$

$\longrightarrow \sum \beta_i \geq 2^c + 2^{c-1} = (1.5)2^c$

Remark: This bound isn't neccessarily true in general.

E.g. It says the last betti # is ≥ 2.

$\{1, 5, 5, 1\}$ is a Betti sequence for a ht 3 ideal in 3 variables. (Artinian)

→ It is also a resolution of a monomial ideal of ht 3 in 5 variables.

Corollary:

If I is monomial $c = n$ and S/I not a CI

then $\sum \beta_i(S/I) \geq 2^c + 2^{c-1}$

$$= (1.5) \cdot 2^c$$

Thm: (Surprisingly) this bound for $\sum \beta_i$ holds

→ For general I provided $c \leq 4$.
(Charalambous - Evans - Miller)

→ For monomial ideals of arbitrary height
(B - Seiner 2018)

New Results (j. with Derrick Wigglesworth)

Theorem (Erman 2009) (Low Regularity)
Suppose that $I \subset S$ has
- $\mathrm{ht}\, I = c$
- smallest degree generator in degree a
- $\mathrm{reg}(S/I) \leq 2a-2$

Then $\beta_i(S/I) \geq \binom{c}{i}$ $\forall i$

(Curves of high degree, Toric Surfaces, ...)

Question (Huneke)

Could one relax the regularity assumption to show (the weaker) bound

$$\sum \beta_i \geq 2^c$$

Answer: no

Question: With the low regularity hypothesis could one prove

$$\sum \beta_i \geq (1.5)2^c$$

Answer: yes!

Erman $\beta_i \geq \binom{c}{i}$

Theorem (B-Wigglesworth) (Low Regularity)

For all $c \geq 3$, if $\mathrm{ht}\, I = c$, $\mathrm{reg}(S/I) \leq 2a-2$

then $$\sum \beta_i(S/I) \geq (1.5)2^c$$

→ (Note the regularity assumption prohibits S/I from being a CI)

Why is $\sum \beta_i \geq 2^c + 2^{c-1} = (1.5)2^c$

Erman : $\boxed{\beta_i \geq \binom{c}{i}}$ $\boxed{\text{Known}}$

Try : $\beta_i \geq \binom{c}{i} + \binom{c-1}{i-1}$ ✗

$\beta_i \geq (1.5)\binom{c}{i}$ ✗

... First half of β_i are $\boxed{\geq 2\binom{c}{i}}$

Thm: (B-Wigglesworth)

If $ht\, I = c$, $reg(S/I) \leq 2a-2$ then

if $c \geq 9$ and $2 \leq i \leq \lceil \frac{c}{2} \rceil$

then $\beta_i(S/I) \geq 2\binom{c}{i}$

→ Cor: If $ht\, I = c \geq 9$ and
I is generated by $2c-1$ quadrics $(a=2)$
then $reg(S/I) > 2(2)-2 = 2$.

$\left(\text{same conclusion if e.g. } \beta_4(S/I) < 2\binom{c}{4}\right)$

This is giving some relationship between the first half of the betti numbers of S/I and the regularity of S/I.

Why is this true?

Erman's approach was to use Boij-Söderberg Theory to bound β_i.

Idea:

$$\beta\text{-Table of } S/I = \sum_{\underline{d} \in A} \overset{\in \mathbb{Q}_{\geq 0}}{\lambda_{\underline{d}}} \underbrace{\left(\beta\text{-table of a pure diagram } \underline{d}\right)}_{\text{these are known}}$$

Sufficient to show:

$$\beta_i(\text{pure diagram } \underline{d}) \geq 2 \cdot \binom{c}{i}$$

$$\frac{\beta_i(\text{pure diagram } \underline{d})}{\binom{c}{i}} \geq 2$$

(i)

Rational function of $n+1$ variables

$\vdots$

<u>Final</u> <u>Reduction</u>: Show for relevant inputs that an explicit rational function of 5 variables is ≥ 2.

$i \leq \lceil \frac{c}{2} \rceil \quad c \geq 9$

$$F(\underset{}{c}, \underset{\text{deg of gen}}{a}, \overset{\beta_i}{i}, \underbrace{b, e}_{\text{regularity}}) \geq 2$$

We have a video where we "animate" the proof of this bound. (In the video, $c = n$)

$$F(a,b,e,n,i) = \frac{(a)\cdots(a+(i-2))}{(b+1)\cdots(b+(i-1))} \, \frac{(n+1)\cdots(n+a+b+e-1)}{(i+1)\cdots(i+a+b+e-1)} \, \frac{e!}{(n-i+1)\cdots((n-i)+e)}.$$

- I has height c in a polynomial ring

	$c \le 4$	$c \geq 5$	Monomial	low regularity
$\beta_i \geq \binom{c}{i}$	✓ easy	?	✓ easy	(2009 Erman) $\Downarrow$
$\sum \beta_i \geq 2^c$	✓ easy	$c=5$ ('93 Avramov-Buchweitz) all c (char $k \neq 2$) ('18 Walker)	$\Downarrow$	$\Downarrow$

- Assume I has height c in $k[x_1, \dots, x_n]$

  → And S/I is not a CI

	$c \le 4$	$c \geq 5$	Monomial: Artinian ($c=n$)	Monomial: Non-Artinian ($c<n$)	low regularity
$\beta_i \geq \binom{c}{i} + \binom{c-1}{i-1}$	False	False	('91 CE) $\Downarrow$	False	False
$\sum \beta_i \geq (1.5) 2^c$	('90 CEM) Classification of Tor Algebras	?	$\Downarrow$	('18 B-Seiner) Delicate Splitting Argument	yes $\forall c$. $\Uparrow$ + check $\sum \beta_i$ for $5 \le c \le 8$
First half of $\beta_i \geq 2\binom{c}{i}$	False	False	False	False	True for $c \geq 9$

$i = 2, \ldots, \lceil \frac{c}{2} \rceil$

$\operatorname{reg}(S/I) < 2a - 2$ $a =$ deg gens.