Symbolic Powers, Interpolation and related problems

Part 1: Interpolation, Alexander-Hirschowitz thm, some open problems) based on [Há-M '21]

Part 2: Some recent results) [Fouli-M-Xie '15], [M '20]

Notation
- (·) $k = \mathbb{C}$,
- (·) $R = \mathbb{C}[x_0, \dots, x_n]$
- (·) $X = \{P_1, \dots, P_r\} = r$ simple points in $\mathbb{P}^n$
- (·) $I_X = \wp_1 \cap \dots \cap \wp_r$ = defining ideal of X

If $J \subseteq R$ is a homogeneous ideal:
- (·) $[R/J]_d = \langle f \in R/J \mid f$ is homog. of degree $d \rangle$
- (·) $H_{R/J}(d) = \dim_{\mathbb{C}} [R/J]_d$ = Hilbert function of R/J in degree d
- (·) $\alpha(J) = \min\{t \mid \exists\, 0 \neq f \in [J]_d\}$ = initial degree of J

(Hermite, homogeneous, uniform) Interpolation Problems: Fix $X \subseteq \mathbb{P}^n$ (pts), $m \in \mathbb{Z}_+$, determine information about all hypersurfaces in $\mathbb{P}^n$ passing through each P_i at least m times

Commutative Algebra: $f = 0$ passes through $X \iff f \in I_X$

Zariski-Nagata: $\underbrace{f = 0 \text{ passes through } X \geq m \text{ times}} \iff f \in I_X^{(m)} := \wp_1^m \cap \dots \cap \wp_r^m = m^{th}$ symbolic power of I_X

(ie. $\frac{\partial f}{\partial x^{\underline{\alpha}}}$ passes through X $\forall \underline{\alpha} \in \mathbb{N}_0^{n+1}$, $|\underline{\alpha}| = m-1$)

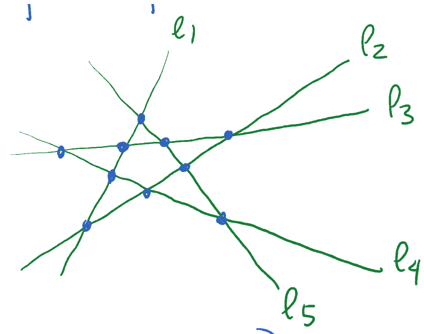
Interpolation problems: Fix $X \subseteq \mathbb{P}^n$ pts, deduce info about $I_X^{(m)}$ (eg. $\alpha(I_X^{(m)})$ or $H_{R/I_X^{(m)}}(d)$)

First observation: The geometry of X matters!

Eg (*) $X = 10$ "random" pts $\subseteq \mathbb{P}^2 \Rightarrow \alpha(I_X) = 4$, $\boxed{\alpha(I_X^{(2)}) = 7}$

(*) X = a star configuration of 10 pts $\subseteq \mathbb{P}^2$, ie.

(*) $X =$ a star configuration:



$\Rightarrow \alpha(I_X) = 4 \quad (\alpha(I_X) > 3$ by Bezout$)$

and $\boxed{\alpha(I_X^{(2)}) = 5} \quad (\ell_1 \cdots \ell_5 \in I_X^{(2)})$

"random" pts $\longleftrightarrow$ general pts (dense, open Zariski subset)

Thm 1: $X = r$ general pts $\subseteq \mathbb{P}^n \Rightarrow \boxed{H_{R/I_X}(d) = \min\left\{\binom{n+d}{n}, r\right\}}, \forall d \geq 1$

$\Rightarrow \alpha(I_X) = \min\left\{d \,\middle|\, \binom{n+d}{n} > r\right\}$

($\Rightarrow H_{R/I_X}$ is as simple as possible.
Eg. if $X = 24$ gen'l pts $\subseteq \mathbb{P}^3 \Rightarrow H_{R/I_X}$: 1 4 10 20 24 24 24 $\rightarrow$)

Thm 2: $X = r$ pts $\subseteq \mathbb{P}^n$, $m \in \mathbb{Z}_+$ $\Rightarrow H_{R/I_X^{(m)}}(d) \leq \min\left\{\binom{n+d}{n}, r \cdot \binom{n+m-1}{n}\right\} \quad \forall d \in \mathbb{Z}_+$

Pf: let $F \in [R]_d$, write $F = \sum_{i=1}^{N} c_i M_i$ $\quad \left(\begin{array}{l} N := \binom{n+d}{n} \\ M_1, \ldots, M_N = \text{all monom. of degree } d \\ c_1, \ldots, c_N \in \mathbb{C} \end{array}\right)$

$$F \in I_X^{(m)} \overset{ZN}{\Longleftrightarrow} \begin{cases} \left(\frac{\partial F}{\partial \underline{x}^{\underline{\alpha}}}\right)(P_1) = 0 \\ \vdots \\ \left(\frac{\partial F}{\partial \underline{x}^{\underline{\alpha}}}\right)(P_r) = 0 \end{cases} \quad \forall \underline{\alpha} \in \mathbb{N}_0^{n+1}, \ |\underline{\alpha}| = m-1$$

$\Rightarrow$ hom. linear system of $r \cdot \binom{n+m-1}{n}$ eq'ns in N variables (the c_i's)

$H_{R/I_X^{(m)}}(d) = N - H_{I_X^{(m)}}(d) = rk(\quad) \leq \min\left\{\binom{n+d}{n}, r \cdot \binom{n+m-1}{n}\right\}$ ($\Rightarrow$ thm 2)

("=" holds if $m=1$ and $X =$ general $\Rightarrow$ thm 1)

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Def (·) (mX or) $I_X^{(m)}$ has expected dim. in degree d $\Leftrightarrow$ $H_{R/I_X^{(m)}}(d) = \min\left\{\binom{n+d}{n},\ r\cdot\binom{n+m-1}{n}\right\}$

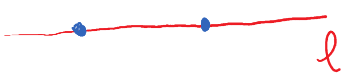
(·) X is AH(d) if $I_X^{(2)}$ has expected dimension in degree d.

Rmk: X is AH(d) $\overset{\text{def}}{\Longleftrightarrow}$ $H_{R/I_X^{(2)}}(d) = \min\left\{\binom{n+d}{n},\ r(n+1)\right\}$

$$\Longleftrightarrow \dim \underbrace{\sigma_r}(V_d^n) = \min\underbrace{\left\{\binom{n+d}{n}-1,\ r\cdot(n+1)-1\right\}}$$

ie. the r^{th} secant variety of the image of d^{th} Veronese $\mathbb{P}^n \to \mathbb{P}^{\binom{n+d}{n}-1}$ has the expected dimension

E.g.: $X = 2$ pts $\subseteq \mathbb{P}^2$. X is AH(2) $\Leftrightarrow$ $H_{R/I_X^{(2)}}(2) = \min\left\{\binom{2+2}{2}, 2(3)\right\} = 6 = H_R(2)$



$\Leftrightarrow \nexists$ quadric in $I_X^{(2)}$

Since $\ell^2 \in I_X^{(2)} \Rightarrow X$ is not AH(2).

However X is AH(3).

Thm (Alexander-Hirschowitz '95): $X = r$ gen'l pts $\subseteq \mathbb{P}^n$, then

X is not AH(d) $\Leftrightarrow$ one of these 3 exceptions:

(i) $2 \le r \le n$, $d = 2$

(ii) $r = 5$ in $\mathbb{P}^2$
$r = 9$ in $\mathbb{P}^3$ $\quad\Big\}\ d = 4$
$r = 14$ in $\mathbb{P}^4$

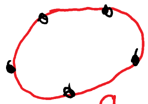
(iii) $r = 7$ in $\mathbb{P}^4$, $d = 3$

$2 \le n \le 4$

Pf: "$\Leftarrow$" (i) as above. $\quad \nexists$ … in $I^{(2)}$

Pf: "$\Leftarrow$" (i) as above

$2 \le n \le 4$

(ii)  q $\quad X$ is $AH(4) \Leftrightarrow \not\exists$ quartic in $I_X^{(2)}$

Since $q^2 \in I_X^{(2)} \Rightarrow X$ is not $AH(4)$

(iii) X is $AH(3) \Leftrightarrow \not\exists$ cubic in $I_X^{(2)}$

Castelnuovo: $\exists$ $\mathcal{C}$ rat'nal normal curve through X

$$I_{\mathcal{C}} = I_2 \begin{bmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \end{bmatrix}$$

$$f := \det[\quad] \in I_{\mathcal{C}}^{(2)} \subseteq I_X^{(2)}$$

$\Rightarrow X$ is not $AH(3)$

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"$\Rightarrow$" Ingredients: ($\cdot$) r gen'l pts have $AH(d) \Leftrightarrow \exists$ one set X_0 of r pts that is $AH(d)$

($\cdot$) Fix n, d. Then

X is $AH(d)$ $\forall r \Leftrightarrow X$ is $AH(d)$ for

$$\left\lfloor \frac{\binom{n+d}{n}}{n+1} \right\rfloor \le r \le \left\lceil \frac{\binom{n+d}{n}}{n+1} \right\rceil$$

($\cdot$) Thm (Main inductive argument): Fix $n \ge 2, d \ge 4$

Let q, ε be s.t. $r(n+1) - \binom{n+d-1}{n} = qn + \varepsilon \qquad 0 \le \varepsilon < n$

If (i) r gen'l dbl pts in $\mathbb{P}^{n-1}$ is $AH(d)$, and

(ii) $r-q$ " " " $\mathbb{P}^n$ is $AH(d-1)$, and

(iii) $r-q-\varepsilon$ " " " $\mathbb{P}^n$ is $AH(d-2)$

$\Rightarrow r$ gen'l dbl pts in $\mathbb{P}^n$ is $AH(d)$.

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Open Problems: ($\cdot$) $\mu(I_X^{(2)}) = ?$

$(\mu(M) = $ min'l # generators$)$

($\mu(I) = \min \#$ of generators)

Open Problems: (•) $\mu(I_X^{(2)}) = ?$

$\mu(I_X^{(m)}) = ?$

(•) $\operatorname{sdef}(I_X, m) = \mu(I_X^{(m)}/I_X^{m}) = ?$

(•) Betti table of $I_X^{(2)}$?

(•) Betti table of I_X ?

(•) $H_{R/I_X^{(m)}}$ for $X \subseteq \mathbb{P}^2$?

<u>SHGH Conj</u>: $X = r$ gen'l pts $\subseteq \mathbb{P}^2$, then

$I_X^{(m)}$ does not have exp. dim. in degree d

$\Longleftrightarrow \exists F$ irred, exceptional for X s.t. $[I_X^{(m)}]_d \subseteq (F)$.

(•) $\alpha(I_X^{(m)}) = ?$ Hard (even for pts in $\mathbb{P}^2$)

<u>Nagata's Conj '58</u>: $X = r \geq 10$ gen'l pts $\subseteq \mathbb{P}^2$

$\Rightarrow \alpha(I_X^{(m)}) > m \cdot \sqrt{r} \quad \forall m \geq 2$

$\left(\Rightarrow \dfrac{\alpha(I_X^{(m)})}{m} > \sqrt{r} \quad \forall m\right)$

<u>Known</u>: $r =$ perfect square (Nagata)

<u>Biran '98</u>: Nagata's Conj. $\Longleftrightarrow$ symplectic packing problem.

(•) weaker estimates on $\alpha(I_X^{(m)})$? (or $\dfrac{\alpha(I_X^{(m)})}{m}$)

<u>Chudnovsky's Conj '79</u>: $\dfrac{\alpha(I_X^{(m)})}{m} \geq \dfrac{\alpha(I_X) + n - 1}{n}$, $\forall m \geq 1$

Part 2a: some recent results.

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Nagata's Conj: $X = r \geq 10$ gen'l pts $\subseteq \mathbb{P}^2 \Rightarrow \frac{\alpha(I_X^{(m)})}{m} > \sqrt{r}$, $\forall m > 1$

Weaker bounds on $\alpha(I_X^{(m)})$ (or $\frac{\alpha(I_X^{(m)})}{m}$) ?

Waldschmidt, Skoda '77: $\forall X \subseteq \mathbb{P}^n$ pts, $\frac{\alpha(I_X^{(m)})}{m} \geq \frac{\alpha(I_X)}{n}$ $\forall m \geq 1$

($\exists$ simple pf using $I^{(ht)} \subseteq I^t$ $\forall t \geq 1$ where $I = \sqrt{I}$ and $h = \max\{ ht(p) \mid p \in Ass(R/I)\}$)

[Ein-Lazarsfeld-Smith '01] $R = \mathbb{C}[\underline{x}]$

[Hochster-Huneke '02] $R = k[\underline{x}]$ any k

[Ma-Schwede '17] $R =$ any regular ring

CC = Chudnovsky's Conj '79: $\frac{\alpha(I_X^{(m)})}{m} \geq \frac{\alpha(I_X) + n - 1}{n}$, $\forall m \geq 1$

Esnault-Viewheg '83: $\frac{\alpha(I_X^{(m)})}{m} \geq \frac{\alpha(I_X) + \varepsilon}{n}$ $\forall m \geq 1$ for some $\varepsilon \geq 1$ ($\Rightarrow \mathbb{P}^2$)

Dumnicki '13: CC holds for general pts $\subseteq \mathbb{P}^3$

Thm: (a) [Fouli-M-Xie '15] CC holds for very general pts $\subseteq \mathbb{P}^n$

(b) [Bisui-Grifo-Hà-Nguyen '20] CC holds for $r \geq 4^n$ general pts $\subseteq \mathbb{P}^n$ ($\geq 2^n$ if $n \geq 9$)

(c) [BGHN '20] X general $\subseteq \mathbb{P}^n \Rightarrow \frac{\alpha(I_X^{(m)})}{m} \geq \frac{\alpha(I_X) + n - 2}{n}$ $\forall m$

Pf: Enough to prove each of them for generic points

(a) Reduce to the case of $\binom{s}{n}$ generic pts ($\forall s \geq n$)

" " " of showing $\exists$ of one set of $\binom{s}{n}$ pts for which CC holds.

Fix $\mathcal{J} = \{l_1, \dots, l_s\}$ hyperplanes meeting properly

The <u>star configuration</u> of $\mathcal{J}$ is

$$I := \bigcap_{1 \le i_1 < \dots < i_n \le s} (l_{i_1}, \dots, l_{i_n}) = \text{ideal of } \binom{s}{n} \text{ pts}$$

$$\frac{\alpha(I^{(nt)})}{nt} = \frac{s \cdot t}{nt} = \frac{s}{n} = \frac{\alpha(I) + n - 1}{n} \qquad \forall t \ge 1$$

$(l_1 \cdots l_s)^t$

$\Rightarrow$ CC holds for star config.

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Part 2b.

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Challenging to determine $\alpha(I_X^{(m)})$, $\mu(I_X^{(m)})$, $\mathrm{sdef}(I_X, m)$, $H_{R/I_X^{(m)}}(d)$, Betti table of $I_X^{(m)}$ when X = general set of pts $\subseteq \mathbb{P}^n$

What can we say for special X?

Def: Let $1 \le c < s$, $\mathcal{F} = \{F_1, \dots, F_s\}$ forms in R that are $(c+1)$-generic

(ie. $\mathrm{ht}(F_{i_1}, \dots, F_{i_{c+1}}) = c+1 \quad \forall\, 1 \le i_1 < i_2 < \dots < i_{c+1} \le s$)

The star configuration on $\mathcal{F}$ of height c is

$$I_c := \bigcap_{1 \le i_1 < \dots < i_c \le s} (F_{i_1}, \dots, F_{i_c}) \underset{[\text{Park-Shin}]}{=} (F_{j_1} F_{j_2} \cdots F_{j_{s-c+1}} \mid 1 \le j_1 < \dots < j_{s-c+1} \le s)$$

Star configurations of pts $\iff \deg(F_i) = 1, \forall i$

E.g.: $s = 7$ ($\Rightarrow \mathcal{F} = \{F_1, \dots, F_7\}$) and forms are 5-generic $\Rightarrow I_c$ for $c \le 4$

eg. $\boxed{I_4 = (F_1F_2F_3F_4, \dots, F_4F_5F_6F_7)}$

If $\deg(F_i) = 2$ $\forall i$, then Betti tables of $I_4^{(5)}$ and $I_4^{(10)}$ are

$I_4^{(5)}$	0	1	2	3	4	
⋮						
21	77					
22		161				
23			105			
24				20		top strand ✓
⋮						
27	210					
28		609				
29			588			
30				189		1st irreg. ✓
33	105					
34		315				
35			315			
36				105		3
39	35					
40		105				regular ✓
41			105			
42				35		1

$\underbrace{\binom{7}{4-1}}$ 35 (1,3,3,1)

$I_4^{(10)}$	0	1	2	3	4	
37	28					
		42				
			15			top ✓
43	413					
		1092				
			952			(we can explain these #s too)
				273		
49	651					
		1890				
			1827			
				588		1st irreg. strand ✓
55	525					
		1575				
			1575			
				525		15
61	350					
		1050				
			1050			
				350		10
67	210					
		630				
			630			
				210		6
73	105					regular ✓
		315				
			315			
				105		3
79	35					
		105				
			105			
				35		

$35\,(1,3,3,1)$

79 | 35 105 105 35   105   3   1

Thm: $\mathcal{F}=\{F_1,\dots,F_s\}$ be $(c+1)$-generic. Then

[M'20] (a) Minimal Generating sets of $I_c^{(m)}$ and $I_c^{(m)}/I_c^m$ $(m\geq 1)$

[M'20] (b) Formulas for $\mu(I_c^{(m)})$ and $\operatorname{sdef}(I_c,m)$ $(m\geq 1)$

[M'20] (c) [Biermann-DeAlba-Galetto-Murai-Nagel-O'Keefe-Römer-Seceleanu '20]
Formula for Betti table of $I_c^{(m)}$ $(m\geq 1)$

Step 1: Define the normal form of a monomial M in $\mathcal{F}$ (ie $M=F_1^{c_1}\cdots F_s^{c_s}$)

$$\boxed{M=M_1^{a_1}\cdots M_t^{a_t}}$$

(·) M_i = sqfree monomial in $\mathcal{F}$
(·) $\operatorname{supp}(M_1)\supsetneq\operatorname{supp}(M_2)\supsetneq\cdots$
(·) $a_i\in\mathbb{Z}_+$

eg. $\mathcal{F}=\{F_1,\dots,F_7\}$, $\hat{M}=F_1^7F_2^5F_3^4F_4^2F_5^4F_6^5F_7^7=(F_1\cdots F_7)^2(F_1F_2F_3F_5F_6F_7)^2(F_1F_2F_6F_7)(F_1F_7)^2$

Step 2: Define $\operatorname{sdeg}_c(M)=\sum_{i=0}^{t}a_i\cdot\max\{0,\,c-s+|\operatorname{supp}(M_i)|\}$

eg. $\operatorname{sdeg}_4(\hat{M})=2(4)+2(3)+1+0=15$
$\operatorname{sdeg}_2(\hat{M})=2(2)+2(1)+0+0=6$

Step 3: thm 1: $\operatorname{sdeg}_c(M)=t \iff M\in I_c^{(t)}\setminus I_c^{(t+1)}$

Step 4: thm 2: A min'l gen. set of

$I_c^{(m)}$ is $\{M=M_1^{a_1}\cdots M_t^{a_t}\mid \operatorname{sdeg}_c(M)=m,\ |\operatorname{supp}(M_t)|>s-c\}$

$I_c^{(m)}/I_c^m$ is $\{$ "   "   "   " , and $|\operatorname{supp}(M_1)|>s-c+1\}$

Cor: $I_2^{(m)}=\left(\left(F_1\cdots F_s\right)^{a_1}\left(F_1\cdots\hat{F_j}\cdots F_s\right)^{a_2}\mid 1\leq j\leq s,\ \underbrace{2a_1+a_2=m}_{2x_1+x_2=m}\right)$

Thm 3: $\mu(I_c^{(m)})=\sum\left(\sum_{d\in S_p}|O_d|\right)$ where ... sol'n to ... $=m\}$

Thm 3: $\mu(I_c^{(m)}) = \sum_{\substack{B=\{b_1,\dots,b_h\} \\ 1\le b_1<\dots<b_h\le c}}' \left(\sum_{\underline{d}\in S_B}' |O_{\underline{d}}| \right)$ where $S_B := \{(d_1,\dots,d_h)\in \mathbb{Z}_+^h \mid b_1x_1+\dots+b_hx_h = m\}$ sol'n to

$O_{\underline{d}} = S_s$-orbit of $N = N_{b_1}^{d_1}\cdots N_{b_h}^{d_h}$

$\left(N_{b_i} := F_1\cdots F_{s-c+b_i} \right)$

$\mathrm{sdef}(I_c, m) = \mu(I_c^{(m)}) - \binom{s}{c-1}$

$\left(\Rightarrow \text{Explicit formulas } (c,s,m,n) \text{ for } \mu(I_c^{(m)}) \text{ when } c\le 4 \text{ or } m\le 4 \right)$

Step 5: An ideal J has c.i. quotients if $\exists$ min'l gen. set $g_1,\dots,g_u$ of J s.t. $(g_1,\dots,g_i):g_{i+1} =$ c.i. ideal

J has δ-c.i. quotients if each is equigenerated in degree δ

Eg: J has linear quotients $\iff J$ has 1-c.i. quotients.

Step 6: Define a total order on the min'l gen. set of $I_c^{(m)}$ (depends heavily on normal form)

Step 7: Thm 4: $I_c^{(m)}$ has c.i. quotients $(\forall m\ge 1)$

$I_c^{(m)}$ has δ-c.i. quotients if $\deg(F_i)=\delta,\ \forall i$

$(\Rightarrow$ stranded Betti tables$)$

Thm 5: (·) Formula for Betti table of $I_c^{(m)}$

(·) closed formulas (c,s,m,n) for

"regular strands" [(*) last strand $\left(=\binom{s}{c-1}\cdot \text{HF of Artin. c.i. of } (c-1) \text{ quadrics}\right)$
(*) $\ge$ half of Betti table is a multiple of J

(*) 1st irregular strand

(*) Top strand $(c\le 4,\ m\le 11)$