When are multidegrees positive?

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Key takeaways

1) multidegrees provide important information about multi-projective varieties.

2) Thus, results on multidegrees translate to (or imply) relevant statements in other areas of math.

I. Degrees

k a field, $R = k[x_0, \ldots, x_n]$,

$$\mathbb{P}^n_k = \operatorname{Proj}(R)$$

$$= \left\{ P \in \operatorname{Spec}(R) \;\middle|\; \begin{array}{l} P \text{ hom.} \\ \text{and } P \nsupseteq (x_0, \ldots, x_n) \end{array} \right\}$$

$$X = \operatorname{Proj}(R/I) \subseteq \mathbb{P}^n_k \quad \text{subvariety:}$$

I hom. prime.

$$d := \dim X = \dim R/I - 1.$$

k alg. closed:

X^{0}

$$\deg(X) = \#\left(X \cap H_1 \cap \cdots \cap H_d\right) < \infty$$

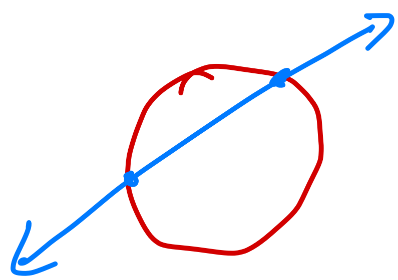
H_i a <u>general</u> hyperplane

Ex: · $d=0 \Rightarrow \deg(X) = \# X$

· $X = V(f)$ hypersurface

$\Rightarrow \deg(X) = \deg(f)$.

e.g. $\deg\left(V\left(x_1^2+x_2^2-x_0^2\right)\right) = 2$



k any field!

Chow ring: $A^*(\mathbb{P}^m_k) = \dfrac{\mathbb{Z}[H]}{(H^{m+1})}$

H: class of a hyperplane.

$[X] = \deg(X)\, H^{m-d}$

Hilbert polynomial :

$$S = R/I = \bigoplus_{n \geq 0} S_n$$

$$\Rightarrow \dim_K S_n \overset{n \gg 0}{=} \sum_{i=0}^{d} e(i) \binom{n+i}{i}$$

$$=: P_X(n)$$

Hilbert polynomial.

$$\boxed{\deg(X) = e(d)}$$

"degree = multiplicity"

Rmk: $\deg(X)$ depends on the embedding $X \subseteq \mathbb{P}_k^m$.

Ex:

$$\Delta_{m,e} : \mathbb{P}_k^m \longrightarrow \mathbb{P}_k^{\binom{m+e}{e}-1}$$

$$(x_0 : \cdots : x_m) \longmapsto \left(x_0^{i_0} x_1^{i_1} \cdots x_m^{i_m} \right)_{i_0 + \cdots + i_m = e}$$

Veronese embedding

$$\deg(\mathbb{P}_k^m) = 1, \quad \deg\left(\Delta_{m,e}(\mathbb{P}_k^m)\right) = e^m.$$

Sometimes varieties come embedded in products of proj. spaces.

(Graphs, blowups, Incidence correspondence, ...)

We need "multi"-version of degree.

II. Multidegrees.

$$R = K[x_{1,0}, \dots, x_{1,m_1}] \otimes \cdots \otimes K[x_{p,0}, \dots, x_{p,m_p}]$$

$$\eta = (x_{1,0}, \dots, x_{1,m_1}) \cdots (x_{p,0}, \dots, x_{p,m_p})$$

$$\mathbb{P} = \mathbb{P}_K^{m_1} \times \cdots \times \mathbb{P}_K^{m_p} = \text{Multiproj}(R)$$

$$= \left\{ P \in \text{Spec}(R) \,\middle|\, \begin{array}{l} P \text{ multihom,} \\ \eta \not\subseteq P \end{array} \right\}$$

$$X = \text{Multiproj}(R/I) \subseteq \mathbb{P} \quad \text{variety,}$$

I multihom. prime.

$$d := \dim X = \dim R/I - p$$

k alg. closed (van der Waerden, '29)

$$\vec{n} = (n_1, \dots, n_p) \in \mathbb{N}^p, \quad |\vec{n}| = n_1 + \dots + n_p = d$$

$$\deg^{\vec{n}}(X) = \#\left(X \cap L_1 \times \dots \times L_p\right) < \infty$$

$L_i \subseteq \mathbb{P}_k^{m_i}$ general linear subspace of dim $m_i - n_i$

multidegree of X of type $\vec{n}$

$$L_i = \bigcap_{j=1}^{n_i} H_{i,j}.$$

Chow ring: (k any field)

$$A^*(\mathbb{P}) = \frac{\mathbb{Z}[H_1, \ldots, H_p]}{(H_1^{m_1+1}, \ldots, H_p^{m_p+1})}$$

H_i: inverse image of hyperplane of $\mathbb{P}_k^{m_i}$ under the nat. projection

$$\pi_i : \mathbb{P} \longrightarrow \mathbb{P}_k^{m_i}$$

$$[X] = \sum_{\substack{|\vec{n}|=d \\ 0 \le n_i \le m_i}} \deg^{\vec{n}}(X)\, H_1^{m_1-n_1} \cdots H_p^{m_p-n_p}$$

$$\in A^*(\mathbb{P}).$$

Hilbert polynomial:

$$S = R/I = \bigoplus_{\vec{v} \in \mathbb{N}^p} S_{\vec{v}}$$

$$\dim_K (S_{\vec{v}}) \overset{\vec{v} \gg 0}{=} \sum_{|\vec{n}| \leq d} e_{\vec{n}}(X) \binom{v_1+n_1}{n_1} \cdots \binom{v_p+n_p}{n_p}$$

$|\vec{n}| = d$, mixed multiplicity.

$$\boxed{\deg^{\vec{n}}(X) = e_{\vec{n}}(X)}$$

"multidegrees = mixed multiplicity"

Why multidegrees?

1) Intersection theory (van der Waerden, '29)

2) Mixed multiplicity of ideals (Bhattacharya, '57)

3) Mult. Sequence (AM, '97; PTUV, '20)

4) Schubert polynomials (Knutson-Miller, '05)

5) Projective degrees of rational maps.

6) Mixed volumes.

7) Algebraic statistics $\begin{pmatrix} \text{MSU2, '16} \\ \text{Huh, '13} \end{pmatrix}$

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8) Cartwright - Sturmfels ideals $\begin{pmatrix} \text{C, DN, G,} \\ \text{'18} \end{pmatrix}$

$\vdots$

A Toric Example

$$A = \begin{bmatrix} 1 & 1 & 1 & 2 & 1 & 0 \\ 2 & 2 & 2 & 1 & 0 & 1 \end{bmatrix}$$

$$\phi_A = (\mathbb{C}^*)^2 \longrightarrow \mathbb{A}^6_{\mathbb{C}}$$

$$(t_1, t_2) \longmapsto (t_1 t_2^2,\ t_1 t_2^2,\ t_1 t_2^2,\ t_1^2 t_2,\ t_1,\ t_2)$$

Affine toric variety

$$X = \overline{\operatorname{Im}(\phi_A)} \subseteq \mathbb{P}^2_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}} \overset{!!}{=} \mathbb{P}$$

multi-projective closure.

$$I = I(X)$$

M2: multidegree I

$$[X] = \overset{\deg^{(0,0,2)}(X)}{1 H_1^2 H_2^2} + \overset{\deg^{(0,1,1)}(X)}{3 H_1^2 H_2 H_3}$$

$$+ \overset{\deg^{(0,2,0)}(X)}{3 H_1^2 H_3^2} + \overset{\deg^{(1,0,1)}(X)}{3 H_1 H_2^2 H_3}$$

$$+ \overset{\deg^{(1,1,0)}(X)}{3 H_1 H_2 H_3^2} \in A^*(\mathbb{P})$$

Notice, the coefficient of

$H_2^2 H_3^2$ is $\emptyset$.

i.e., $\boxed{\deg^{(2,0,0)}(X) = 0}$

Motivating question:

<u>Q:</u> when are multidegrees positive?

i.e., for which $|\vec{n}| = d$, do we

have $\deg^{\vec{n}}(X) \neq 0$?

III. Main result

K $\boxed{\text{any field}}$

$$X \subseteq \mathbb{P} = \mathbb{P}_K^{m_1} \times \cdots \times \mathbb{P}_K^{m_p} \text{ variety,}$$

$$d = \dim X .$$

$$\forall J = \{j_1, \dots, j_n\} \subseteq \{1, \dots, p\} =: [p]$$

$$\pi_J : \mathbb{P} \longrightarrow \mathbb{P}_K^{m_{j_1}} \times \cdots \times \mathbb{P}_K^{m_{j_n}},$$

nat projection.

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Theorem: (Castillo, Cid-Ruiz, Li, M, Zheng, '20)

$$|\vec{n}| = d.$$

$$\deg^{\vec{n}}(X) > 0 \iff n_{j_1} + \dots + n_{j_w} \leq \dim(\Pi_J(X)),$$

$$\forall J \subseteq [p]$$

Toric Example (revisited).

Set $r_J = \dim(\pi_J(X))$.

MZ + Theorem:

$\deg^{\vec{n}}(X) > 0 \iff$

$$\begin{aligned}
n_1 + n_2 + n_3 &= \dim(X) = 2\\
n_1 + n_2 &\le r_{12} = 2\\
n_2 + n_3 &\le r_{23} = 2\\
n_1 + n_3 &\le r_{13} = 2\\
n_1 &\le r_1 = 1\\
n_2 &\le r_2 = 2\\
n_3 &\le r_3 = 2.
\end{aligned}$$

Some history (Bigraded case)

$\boxed{p=2}$

Katz - Mandel - Verma, '94 (Conjecture).

Trung, '02

Huh, '12

Algebraic Version (Mixed multiplicity).

A Artinian local ring.

$$R = \bigoplus_{\vec{v} \in \mathbb{N}^p} R_{\vec{v}}$$ Std. $\mathbb{N}^p$ - graded A-algebra.

(Herrmann, Hyry, Ribbe, Tang, '97)

$\Rightarrow$ R has a Hilb. Polynomial P_R.

Top coefficients: $e_{\vec{n}}(R)$: Mixed multiplicity.

Ass. formula : reduce to the k-algebra case.

IV. Applications

Mixed multiplicities of ideals:

(R, m, k) Noeth local ring,

$I_0, I_1, \dots, I_p \subseteq R$ ideals of pos. ht.

$\sqrt{I_0} = m$.

$$T = \bigoplus_{i_0, \dots, i_p \geq 0} \frac{I_0^{i_0} I_1^{i_1} \cdots I_p^{i_p}}{I_0^{i_0+1} I_1^{i_1} \cdots I_p^{i_p}},$$

T std. $\mathbb{N}^{p+1}$-graded

R/I_0-algebra

H. pol.

$$\deg(P_T) = \dim R - 1$$

Def: $|\vec{n}| = \dim R - 1$

$$e_{\vec{n}}(I_0 | I_1, \dots, I_p) := e_{\vec{n}}(T)$$

Mixed mult. of $I_0, \dots, I_p$.

(Teissier, '73): Milnor numbers.

Theorem (C, CR, L, M, Z)

(R, m) std. graded domain over k,

$|\vec{n}| = \dim R - 1$, $I_0, I_1, \dots, I_p$ are gen. in one degree.

analytic spread.

$$e_{\vec{n}}(I_0 | I_1, \dots, I_p) > 0 \iff$$

$$n_{j_1} + \dots + n_{j_w} \le \ell(I_{j_1} \cdots I_{j_w}) - 1,$$

$$\forall J = \{j_1, \dots, j_w\} \subseteq [p]$$

In particular, if $\ell(I_i) = \dim R$

$$\Rightarrow \quad e_{\vec{n}}(I_0 \mid I_1, \dots, I_p) > 0, \quad \forall \vec{n}.$$

Polymatroids:

$r : 2^{[p]} \longrightarrow \mathbb{Z}_{\geq 0}$, is a <u>rank function</u> if:

1) $r(J_1) \leq r(J_2), \forall J_1 \subseteq J_2$ (non-decreasing)

2) $r(J_1 \cap J_2) + r(J_1 \cup J_2)$
$$\leq r(J_1) + r(J_2),$$
$\forall J_1, J_2$ (submodular)

the set:

$$\mathcal{P} = \left\{ \vec{n} \in \mathbb{N}^p \;\middle|\; \sum_{i \in J} n_i \le r(J), \forall J \subseteq [p], \quad \sum_{i \in [p]} n_i = r([p]) \right\}.$$

is called a polymatroid on the rank function r.

$$M\,\mathrm{supp}(X) := \left\{ |\vec{n}| = d \;\middle|\; \deg^{\vec{n}}(X) > 0 \right\}$$

Theorem (C, ER, L, M, Z)

$r(J) = \dim(\pi_J(X))$ is a rank function, thus $M\,\mathrm{supp}(X)$ is a polymatroid.

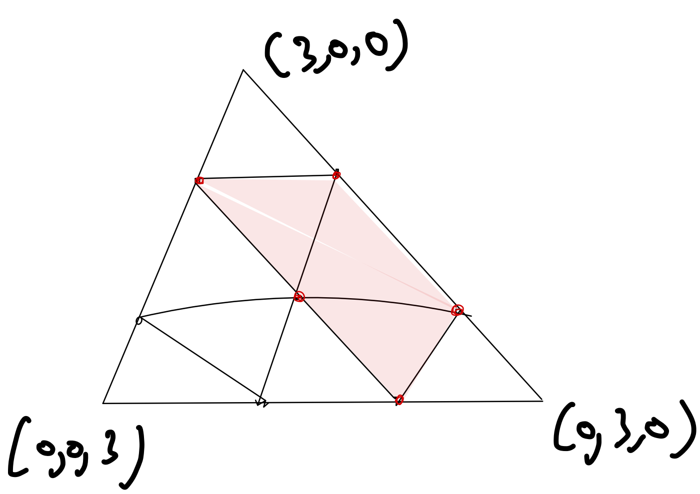
Schubert Polynomials :

S_p : Symmetric group on $[p]$.

for any $\pi \in S_p$, $\mathfrak{S}_\pi$: Schubert Polynomial.

Ex:

$$\mathfrak{S}_{1432} = t_1^2 t_2 + t_1 t_2^2 + t_1^2 t_3 + t_1 t_2 t_3 + t_2^2 t_3$$



(Monical-Tokcan-Yong, '17)

$$q = \mathfrak{C}_\pi = \sum c_{\vec{n}}\, t^{\vec{n}}$$

Conjecture: q has Saturated Newton polytope (SNP), i.e.,

$$\vec{n} \in \mathrm{Newton}(q) \iff c_{\vec{n}} \neq 0.$$

(Fink-Mészáros-St. Dizier, '18)

Conjecture is true ✓.

Our theorem + (Knutson-Miller '04):

Alternative proof. ✓

V. Further questions.

1) (Trung '02) (Bigraded case)

If X is arithmetically CM (ACM)

(i.e., $R/0:_{\mathfrak{a}} 2^\infty$ is CM)

$\Rightarrow$ Trung's result is true.

(Connected in cod. 1 & equidimensional).

Our result is not true.

Q: If X is ACM,

MSupp(X) is a polytope?

Bigraded case.

2) (Huh '12) characterized (up to a multiple) the representable $MSupp(X)$.

<u>Q</u>: Can we characterize the representable $MSupp(X)$ in the multi-projective case?

(Brion, '03), (C, DN, G, '18).

3) $MSupp(X)$ is an invariant of $Hilb^n(\mathbb{P})$ $\rightarrow$ (Haiman – Sturmfels '04)

Convex Geometry $\longleftrightarrow$ Algebraic Geometry

<u>Q</u>: what's the relation between these objects?