Extremal Singularities & Hypersurfaces

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WARM UP: Cubic surfaces

Fix homog cubic $F \in k[x,y,z,w]$ $k = \bar{k}$

$X = \mathbb{V}(F) \subseteq \mathbb{P}^3$ cubic surface

$\left[X \text{ smooth} \iff \left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}, \frac{\partial F}{\partial w} \right\rangle \text{ m-primary } \quad m = \langle x,y,z,w \rangle \right]$

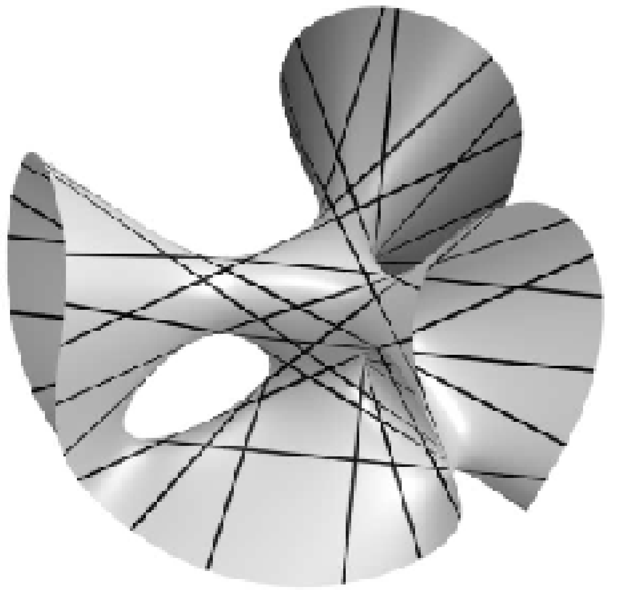
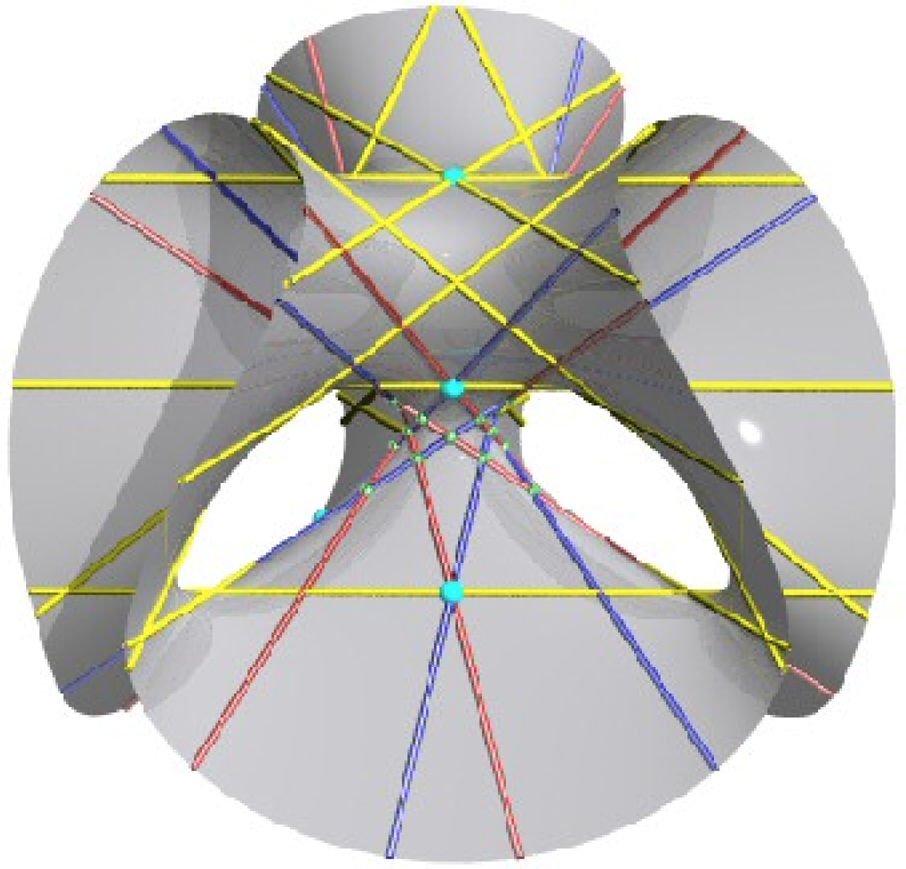


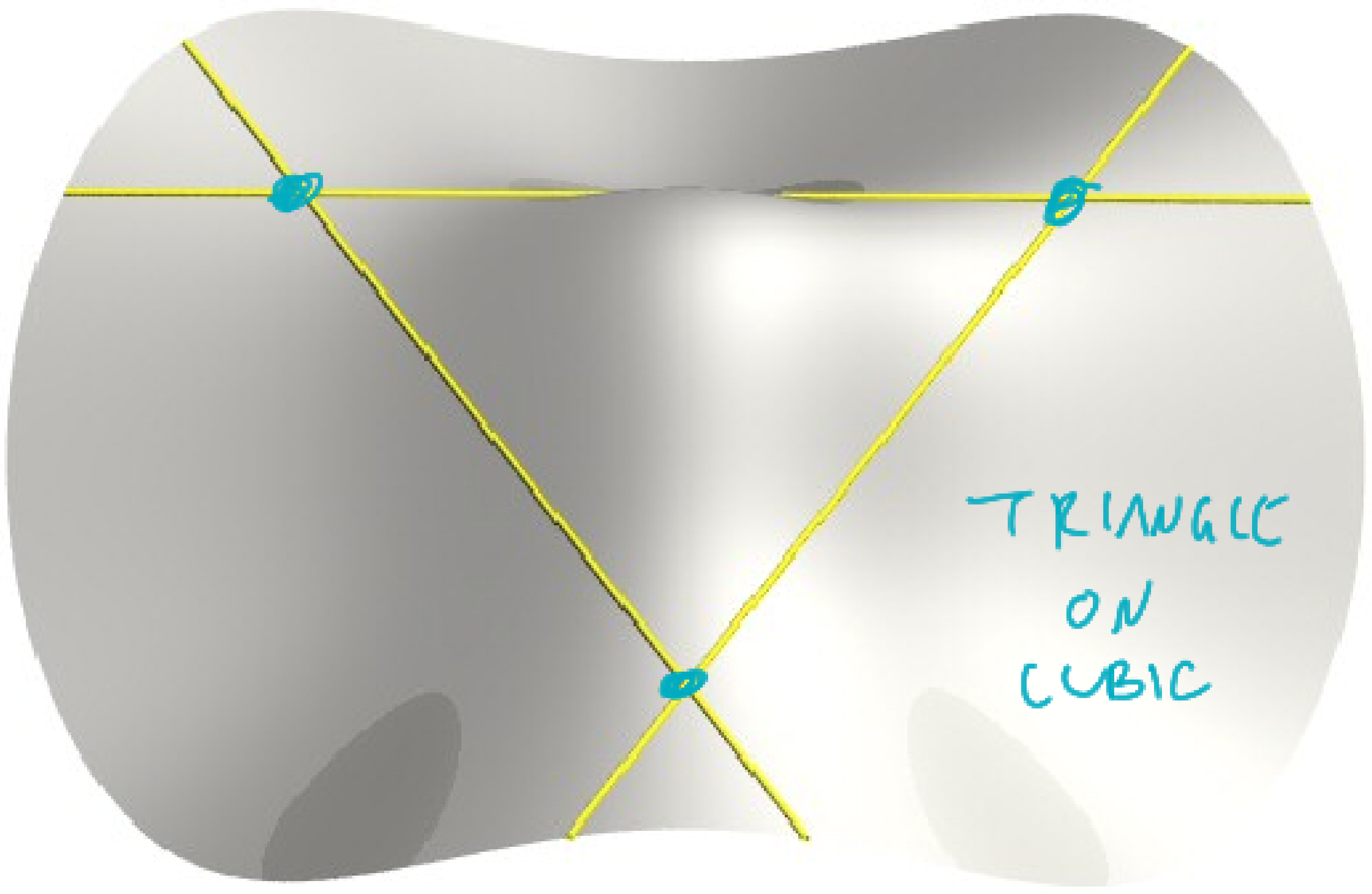
FIGURE 4. Cubic surface with 27 lines (courtesy of Oliver Labs).

FAMOUS Thm (Cayley-Salmon 1849)

Every smooth cubic surface contains EXACTLY 27 lines



Hyperplane sections



$X = \mathbb{V}(F) \subseteq \mathbb{P}^3$ cubic surface

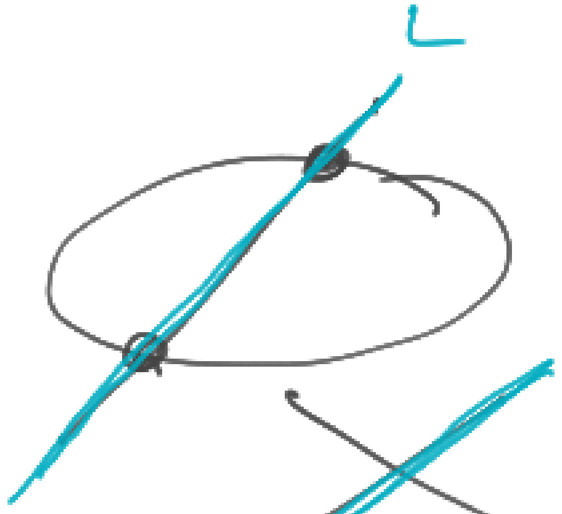
$X \cap H \subseteq H \cong \mathbb{P}^2$ plane cubic curve

$H = \mathbb{V}(ax+by+cz+dw)$

Typically: $X \cap H$ smooth

However: if H is tangent to X at p, $X \cap H$ is singular cubic

- Typically: 

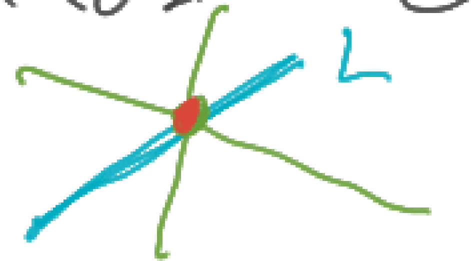
- If H contains a line L $X \cap H$ 

Line and conic

Special case: conic degenerates 

TRI-TANGENT

45 of these

VERY SPECIAL CASE: Eckardt pt. 

TRITANGENT MULT 3 at p

EXAMPLE: FERMAT CUBIC $x^3+y^3+z^3+w^3$ (char $k \neq 3$)

$H = \mathbb{V}(z+w)$ $\quad X \cap H = \mathbb{V}(x^3+y^3+\underbrace{z^3+w^3}_{(z+w)(z^2+wz+w^2)},\ z+w) \subseteq \mathbb{P}^3$

$= \mathbb{V}(x^3+y^3,\ z+w) \subseteq \mathbb{P}^3$

in plane H

$x^3+y^3=0$

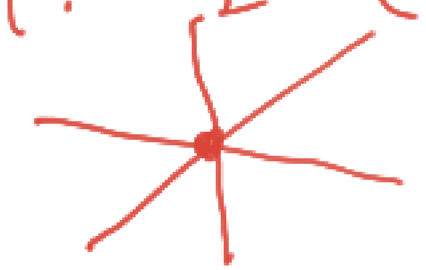
$0=(x+y)(x+\mu)(x+\mu^2 y)$

μ

18 Eckardt pts of this type.

"TRIANGLE-FREE CUBIC SURFACES?"

Q: ARE THERE ANY CUBIC SURFACES $X \subseteq \mathbb{P}^3$ s.t. EVERY TRI-TANGENT PLANE produces a "star"



???

PHILOSOPHY: Geometry of projective hypersurface $\mathbb{V}(F) \subseteq \mathbb{P}^n$

$\updownarrow$

Singularities of affine hypersurface $\operatorname{Spec} \dfrac{k[x_0, \ldots x_n]}{\langle F \rangle}$
(at vertex, m)

$\updownarrow$

Singularities of pair $(k[x_0 \cdots x_n], F)$

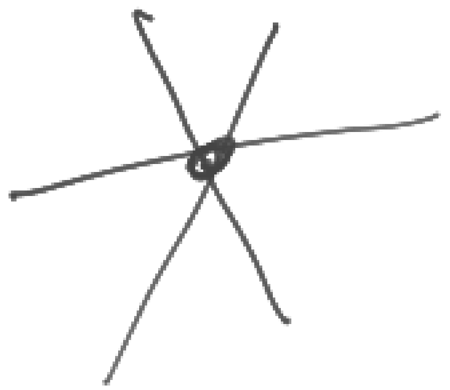
Measuring SINGULARITIES:

$\mathbb{C}$: "log canonical threshold" of F (at m)

char p "F-pure threshold"

THEOREM: Given $F \in k[x,y,z,w]$ smooth cubic $k = \bar{k}$

THE FOLLOWING ARE EQUIVALENT:



1. $X = V(F) \subseteq \mathbb{P}^3$ has NO TRIANGLES
2. The ring $\dfrac{k[x,y,z,w]}{\langle F \rangle}$ is NOT F-pure (NOT Frobenius split)
3. The "F-pure threshold" is SMALLEST POSSIBLE, $1/2$.
4. $\operatorname{char} k = 2$ and $F = x^3 + y^3 + z^3 + w^3$ after linear change of coordinates

(indebted N. Hara)

MEASURING SINGULARITY : MULTIPLICITY

$0 \in V(f)$

$f \in k[x_1,\dots,x_N]$ $\quad f \in m = \langle x_1,\dots,x_N\rangle$

DEF: The multiplicity of f (at m) is MAXIMAL T s.t. $f \in m^T$

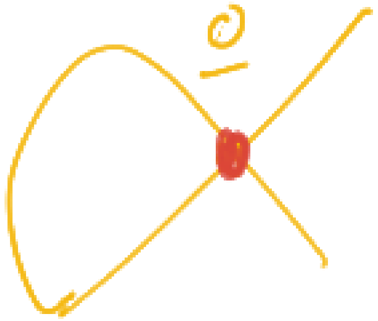
$$\mathrm{mult}_0 f = \sup\{T \mid f \in m^T\}$$

$$f = \underbrace{\sum_{i=1}^{N} \frac{\partial f}{\partial x_i}\Big|_0 x_i}_{\text{linear part}} + \underbrace{\sum_{i,j}^{N} \frac{\partial^2 f}{\partial x_i \partial x_j}\Big|_0 x_i x_j}_{\text{quadratic part}} + H.O.T.$$

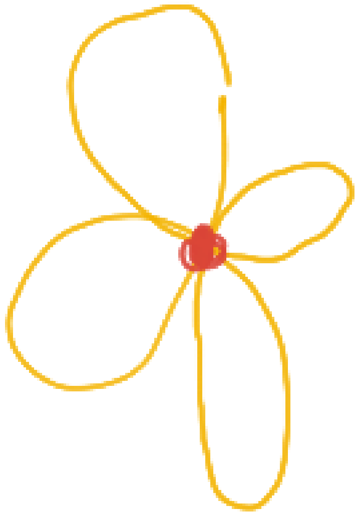
FACTS: • $V(f)$ is smooth at 0 $\iff$ $\mathrm{mult}_m f = 1$

• F homog, $\mathrm{mult}_m F = \text{degree } d$

$\mathrm{mult}_0 f = 1$



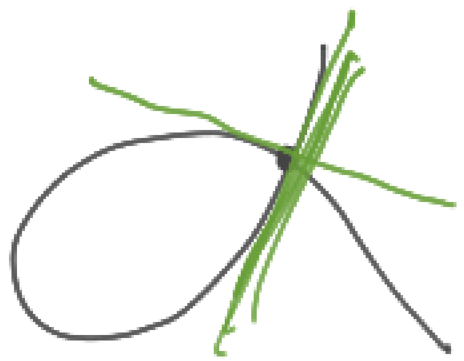
$\mathrm{mult}_0 f = 2$



$\mathrm{mult}_0 (f) \geq 4$

Some degree 2 singularities:

$y^2 - x^2 - x^3$



"Barely singularity"

"Simple normal crossing"

$LCT(y^2 - x^2 - x^3) = 1$

$y^2 - x^3$

Cusp

MORE SINGULARITY

$LCT(y^2 - x^3) =$

$1/2 + 1/3 = 5/6$

$y^2 - x^{17}$

Sharper cusp

Much more singular

$LCT(y^2 - x^{17})$

$1/2 + 1/17$

SMALLER LCT for "MORE SINGULAR pts"

Over $\mathbb{C}$: Log canonical threshold

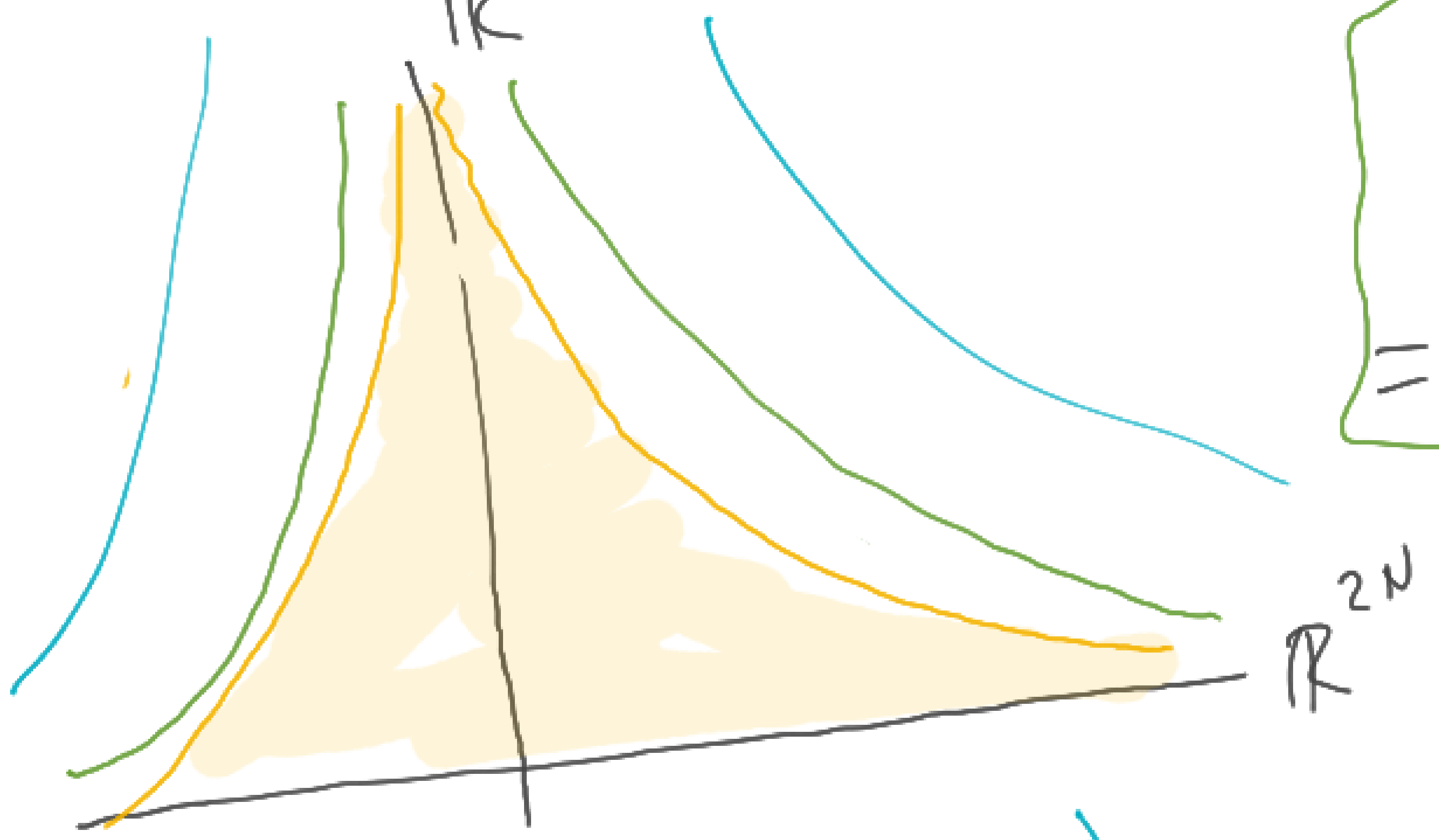
" Looking at "function" $\mathbb{C}^N = \mathbb{R}^{2N} \dashrightarrow \mathbb{R}$

$$\underline{x} \mapsto \frac{1}{|f(x)|}$$

"blowing up" to ∞ as we approach $V(f)$

"blowing up even faster" at singular points of $V(f)$

Cartoon of graph



Log canonical threshold.

DEF: The log canonical of f at 0

$$= \sup\left\{ \lambda \;\middle|\; \int_{B_0(\varepsilon)} \frac{1}{|f(x)|^{2\lambda}} < \infty \right\}$$

$B_0(\varepsilon)$ Ball of radius ε around $\underline{0}$

converges for small λ

PROPERTIES OF LOG CANONICAL THRESHOLD

- $0 < \mathrm{LCT}(f) \leq 1$
- f is smooth at $0 \implies \mathrm{LCT}(f) = 1$
  ($\mathrm{LCT}(f) = 1 \iff f$ is "log canonical")
- $\mathrm{LCT}(f) \in \mathbb{Q}$ (follows from Resolution of Singular)
- Smaller LCT means "more singular".

CHAR p: F-pure Threshold (Hara-Yoshida, Takagi-Watanabe)

$f \in k[x_1 \cdots x_N]$ char $k = p > 0$ $f \in \mathfrak{m} = \langle x_1, \ldots, x_N \rangle$

NOTATION: $\mathfrak{m}^{[p^e]} = \langle x_1^{p^e}, x_2^{p^e}, \ldots, x_N^{p^e} \rangle$ "Frobenius power of $\mathfrak{m}$"

DEF: THE F-pure threshold of f at $\mathfrak{m}$ $= \sup \left\{ \frac{d}{p^e} \;\middle|\; f^d \notin \mathfrak{m}^{[p^e]} \right\} = \operatorname{INF} \left\{ \frac{d}{p^e} \;\middle|\; f^d \in \mathfrak{m}^{[p^e]} \right\}$

Philosophically:

$f^d \notin \mathfrak{m}^{[p^e]} \iff f^{d/p^e} \notin \mathfrak{m}$ (NOT vanishing at 0)

"Frobenius allows fractional powers"

Properties of F-pure Threshold (FPT)

- $0 < FPT_m(f) \leq 1$
- f smooth at $\underline{0}$ $\Longrightarrow$ $FPT(f) = \underline{1}$
  ($FPT(f) = 1 \iff F$ is F-pure)
- $FPT(F) \in \mathbb{Q}$ (Thm of Blickle, Mustata, S–)
- $1/\text{mult}(f) \leq FPT(f)$
- SMALLER FPT means MORE SINGULAR.

EXAMPLE: Consider $f = \underline{y^2} - x^3 \in k[x,y]$ 

when $k = \mathbb{C}$ $\qquad LCT(f) = 5/6$

when char $k = p$

$$FPT(f) = \begin{cases} 1/2 & \text{if } p = 2 \\ 2/3 & \text{if } p = 3 \\ 5/6 - 1/6p & \text{if } p \equiv 2 \bmod 3,\ p \neq 2 \\ \underline{\underline{5/6}} & \text{if } p \equiv 1 \bmod 3 \end{cases}$$

||| smaller

NO MORE SINGULAR than / $\mathbb{C}$

THM: $F \in \mathbb{Z}[x_1, \dots, x_n]$

$$\sup \left\{ FPT_m\left(\begin{matrix} F \bmod p \\ \text{in } \mathbb{F}_p[x_1, \dots, x_n] \end{matrix} \right) \;\middle|\; p \text{ primes} \right\} = LCT\left(\underset{\text{over } \mathbb{C}}{f} \right)$$

What is "the most singular" hypersurface of degree d?

IS there a LOWER BOUND on the F-pure threshold of a HOMOGENEOUS f in terms of its degree?

If so, are there polynomials that achieve this lower Bound?

IF SO, can we classify them?

THM: Let $f \in k[x_1, \dots, x_n]$ homog degree d, REDUCED / $\overline{k}$, char $k = p$

THEN
- $FPT_m(f) \geq \frac{1}{d-1}$
- EQUALITY HOLDS $\iff$ $d = p^e + 1$ for some $e \in \mathbb{N}$ AND $f \in m^{[p^e]}$

$\iff f = x_1^{p^e} L_1 + \dots + x_n^{p^e} L_n$ where L_i are linear forms

CLASSIFICATION OF FROBENIUS FORMS / $k=\bar{k}$

DEF: A Frobenius Form is homog pol'g of form

$$f = x_1^{p^e} L_1 + \cdots + x_N^{p^e} L_N = \underbrace{[x_1^{p^e} \cdots x_N^{p^e}]}_{1\times N} \underbrace{A}_{N\times N \text{ matrix}/k} \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}}_{N\times 1}$$

where L_i are degree 1

FACT: There are only finitely many up to linear change of coord. in each N and degree p^e+1

ONLY ONE (for each N, p^e+1) defining smooth $\mathbb{V}(F) \subseteq \mathbb{P}^{N-1}$

$(x_1^{p^e} + \cdots + x_N^{p^e})$

$$\left\{\begin{matrix}\text{NON DEGENERATE} \\ \text{Frob form in} \\ N\text{-variables} \\ \text{(Fixed degree)}\end{matrix}\right\} \Big/ \sim \quad \xleftrightarrow{1-1} \quad \text{partitions of } N$$

$1+1+\cdots+1=N$

Outline of Proof: OF LOWER BOUND ON FPT(f)

1. WLOG $k = \overline{k}$
2. Reduces to 2 variable case
   - general hyperplane section of reduced hypersurface $\rightsquigarrow$ still reduced (Bertini-type thm)
   - LEMMA $k[x_1 \ldots x_n] \longrightarrow \dfrac{k[x_1 \ldots x_n]}{\langle \text{LINEAR fcn} \rangle} \cong k[x_1, \ldots, x_{n-1}]$

$$FPT(f) \geq FPT(\overline{f})$$

$\overline{f}$ = image of f in quotient RING.

form in fewer variables

PROOF: $FPT(f) = \text{INF}\left\{ \dfrac{a}{p^e} \;\middle|\; f^a \in \mathfrak{m}^{[p^e]} \right\}$

$$f^a \in \mathfrak{m}^{[p^e]} \Longrightarrow \overline{f}^a \in \overline{\mathfrak{m}}^{[p^e]}$$

SO $\left\{ \dfrac{a}{p^e} \;\middle|\; f^a \in \mathfrak{m}^{[p^e]} \right\} \subseteq \left\{ \dfrac{a}{p^e} \;\middle|\; \overline{f}^a \in \overline{\mathfrak{m}}^{[p^e]} \right\}$

SO INF OF L.H.S $\leq$ INF R.H.S

③ To get the LOWER BOUND $FPT(f) \geq \frac{1}{d-1}$ $\quad f \in k[x,y]$

$$FPT = \sup \left\{ \frac{N}{p^e} \;\middle|\; f^N \in \mathfrak{m}^{[p^e]} \right\}$$

SUFFICE: if $f^N \in \mathfrak{m}^{[p^e]}$, then $\frac{N}{p^e} \geq \frac{1}{d-1}$

SAY $f = xyg$ $\quad g$ has degree $d-2$

SAY $f^N \in \langle x^{p^e}, y^{p^e} \rangle \Rightarrow x^N y^N g^N \in \langle x^{p^e}, y^{p^e} \rangle$

$\Rightarrow g^N \in \langle x^{p^e-N}, y^{p^e-N} \rangle$ (using x,y regular sequence)

Because $g \neq 0$

$$N \cdot \deg g = N(d-2) \geq p^e - N$$

SO $\frac{N}{p^e} \geq \frac{1}{d-1}$. QED

④ Consider f s.t. $FPT(f) = \frac{1}{d-1}$
homog. degree d
in 2 variables

First show:

$\underline{4a}$: degree $f = p^e + 1$ for some e
(Hardest part, estimates, analysis)

NEXT $\underline{4b}$: $\underline{\text{If}}$ $f \notin \mathfrak{m}^{[p^e]}$, then $FPT(f) \geq \frac{1}{p^e} + \left(\frac{1}{p^e}\right)^2$

Putting 4a,b together, gives characterization
of forms of DEGREE WITH SMALLEST
possible F-PT as Frobenius Form.

CUBICS

$F \in k[x_1, \dots, x_n]$ degree 3 $\quad FPT(F) \geq \frac{1}{2}$

EQUALITY holds $\Leftrightarrow$ $p = 2$ and $f \in \mathfrak{m}^{[2]}$

WHEN DOES A CUBIC SURFACE HAVE TRIANGLES?