The compressible Euler equations in a physical vacuum: a comprehensive Eulerian approach

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The compressible Euler equations

Variables: v = gas velocity, ρ = gas density.

System in $\mathbb{R}^{1+d}$, $d \geq 1$:

$$\begin{cases} \rho_t + \nabla(\rho v) = 0 & \text{(conservation of mass)} \\ \rho(v_t + (v \cdot \nabla) v) + \nabla p = 0 & \text{(conservation of momentum)} \end{cases}$$

Cauchy data:

$$\rho(t=0) = \rho_0, \qquad v(t=0) = v_0$$

Constitutive law

$$p = p(\rho).$$

Standard model:

$$p(\rho) = \rho^{\kappa+1}, \qquad \kappa > 0$$

The structure of the equations

Material derivative:

$$D_t = \partial_t + v \cdot \nabla$$

With this notation the system is rewritten as

$$\begin{cases} D_t \rho + \rho \nabla v = 0 \\ \rho D_t v + \nabla p = 0. \end{cases}$$

a) Wave equation for $\rho, \nabla \cdot v$:

$$D_t^2 \rho - \rho \nabla (\rho^{-1} p'(\rho) \nabla \rho) = \rho [(\nabla \cdot v)^2 - Tr(\nabla v)^2]$$

Propagation speed (sound speed):

$$c_s^2 = p'(\rho)$$

b) Transport equation for the vorticity $\omega =$ curl v

$$D_t \omega = - \omega \cdot \nabla v - (\nabla v)^T \omega$$

Vacuum states

Vacuum state: $\rho = 0$

Gas domain $\Omega(t) \subset \mathbb{R}^d$

Free boundary: $\Gamma(t) = \partial\Omega(t)$

Fluid vs. gas:

- Fluid: $\rho \not\to 0$ on Γ.
- Gas: $\rho \to 0$ on Γ.

Acceleration of particles on free boundary: $a = -\nabla c_s^2$.

Physical vacuum:

$$c_s^2(x) \approx d(x, \Gamma)$$

- Stable evolution mode
- Nontrivial dynamics for the free boundary

The well-posedness question

Lagrangian setting: Parametrize by initial particle position. Particles are fixed, geometry is changing.

Earlier work: Existence and uniqueness in high regularity spaces.

- Coutand-Lindblad-Skholler
- Jang-Masmoudi
- Coutand-Skholler

Eulerian setting: Use euclidean coordinates. Particles are moving, geometry is fixed.

Our objectives:

1. Redevelop theory fully in Eulerian setting.
2. Obtain sharp results in terms of regularity.
3. Provide a complete theory, including difference bounds, continuous dependence.
4. Framework that applies to the relativistic case.

The good variables

Recall:

$$p(\rho) = \rho^{\kappa+1}, \qquad c_s^2 = (\kappa+1)\rho^\kappa$$

New variable

$$r = \frac{\kappa+1}{\kappa}\rho^\kappa \quad \Rightarrow \quad c_s^2 = \kappa r$$

Equation for good variables (r, v):

$$\begin{cases} D_t r + \kappa r \nabla v = 0 \\ D_t v + \nabla r = 0. \end{cases}$$

Conserved energy:

$$E = \int r^{\frac{1-\kappa}{\kappa}} \left(r^2 + \frac{\kappa+1}{2} r v^2 \right) dx$$

Sobolev spaces

Energy space $\mathcal{H}$:

$$\|(s,w)\|_{\mathcal{H}}^2 = \int r^{\frac{1-\kappa}{\kappa}} \left(|s|^2 + \kappa r |w|^2\right) \, dx$$

Acoustic metric:

$$ds^2 = \frac{1}{r}\, dx^2$$

Higher Sobolev spaces $\mathcal{H}^{2k}$:

$$\|(s,w)\|_{\mathcal{H}^{2k}}^2 = \sum_{|\beta| \leq 2k}^{|\beta|-\alpha \leq k} \|r^\alpha \partial^\beta (s,w)\|_{\mathcal{H}}^2$$

- Noninteger case defined by interpolation.

State space $\mathbf{H}^{2k}$: "infinite dimensional manifold"

$$\mathbf{H}^{2k} = \{(r,v) \mid \ (r,v) \in \mathcal{H}^{2k}, \ |\nabla r| > 0 \text{ on } \Gamma\}$$

Scaling and control norms:

$$(r(t,x), v(t,x)) \to (\lambda^{-2} r(\lambda t, \lambda^2 x), \lambda^{-1} v(\lambda t, \lambda^2 x)).$$

Scaling based counting for *order* of factors in multilinear expressions:

1. r and v have degree -1, respectively $-\frac{1}{2}$.
2. ∇ has order 1 and D_t has order $\frac{1}{2}$.

Critical Sobolev space $\mathbf{H}^{2k_0}$:

$$2k_0 = d + 1 + \frac{1}{\kappa}$$

Control parameters:

$$A = \|\nabla r - N\|_{L^\infty} + \|v\|_{\dot{C}^{\frac{1}{2}}} \qquad (\text{bounded by } \mathbf{H}^{2k_0+})$$

$$B = \|\nabla r\|_{\tilde{C}^{0,\frac{1}{2}}} + \|\nabla v\|_{L^\infty} \qquad (\text{bounded by } \mathbf{H}^{2k_0+1+})$$

where the $\tilde{C}^{0,\frac{1}{2}}$ norm is given by

$$\|f\|_{\tilde{C}^{0,\frac{1}{2}}} = \sup_{x,y\in\Omega_t} \frac{|f(x)-f(y)|}{r(x)^{\frac{1}{2}} + r(y)^{\frac{1}{2}} + |x-y|^{\frac{1}{2}}}.$$

Main results I

Theorem (Uniqueness)

For every Lipschitz initial data (r_0, v_0) *satisfying the nondegeneracy condition* $|\nabla r_0| > 0$ *on* Γ_0*, the solution* (r, v) *is unique in the class*

$$v \in C^1_x, \qquad \nabla r \in \tilde{C}^{0,\frac{1}{2}}_x.$$

Theorem (Well-posedness)

The (r, v) *system is locally well-posed in the space* $\mathbf{H}^{2k}$ *for* $k \in \mathbb{R}$ *with*

$$2k > 2k_0 + 1.$$

Full Hadamard+ quasilinear well-posedness:

- Existence of solutions $(r, v) \in C[0, T; \mathbf{H}^{2k}]$.
- Uniqueness of solutions in a larger class, see the uniqueness Theorem above.
- Weak Lipschitz dependence on the initial data, relative to a new, nonlinear distance functional.
- Continuous dependence of the solutions on the initial data in $\mathbf{H}^{2k}$.

Main results II

Theorem (Energy estimates)

For each integer $k \geq 0$ there exists an energy functional E^{2k} such that:

$$E^{2k}(r,v) \approx \|(r,v)\|^2_{\mathcal{H}^{2k}} \qquad \textit{(coercivity)}$$

$$\frac{d}{dt}E^{2k}(r,v) \lesssim_A B\|(r,v)\|^2_{\mathcal{H}^{2k}} \qquad \textit{(growth bound)}$$

Gronwall (also for noninteger k by interpolation):

$$\|(r,v)(t)\|^2_{\mathcal{H}^{2k}} \lesssim e^{\int_0^T C(A)B(s)\,ds}\|(r,v)(t)(0)\|^2_{\mathcal{H}^{2k}}.$$

Theorem (Continuation)

Let $2k > 2k_0 + 1$. Then the $\mathbf{H}^{2k}$ solutions can be continued for as long as A remains bounded and $B \in L^1_t$.

Same scale as standard continuation results in boundaryless case !

The linearized equation

Linearized variables (s, w)

$$\begin{cases} D_t s + w \cdot \nabla r + \kappa(s\nabla \cdot v + r\nabla \cdot w) = 0 \\ D_t w + (w \cdot \nabla)v + \nabla s = 0. \end{cases}$$

Proposition

Assume that A is bounded and $B \in L^1_t$. Then the linearized equation is well-posed in $\mathcal{H}$, and the following energy estimate holds:

$$\left| \frac{d}{dt} \|(s, w)\|^2_{\mathcal{H}} \right| \lesssim \|\nabla v\|_{L^\infty} \|(s, w)\|^2_{\mathcal{H}}$$

- No boundary conditions are imposed or needed on Γ !
- State space $\mathcal{H}$ is time dependent.
- Equation is imposed in the sense of distributions.
- Proof is by energy estimates and duality.

The second order transition operators

Interpret evolution as second order in time:

$$D_t^2 s \approx L_1 s, \qquad L_1 s = \kappa r \Delta s + \nabla r \cdot \nabla s$$

- L_1 is coercive, self-adjoint in first component of $\mathcal{H}$.

$$D_t^2 w \approx L_2 w, \qquad L_2 w = \kappa \nabla(r \nabla \cdot w) + \nabla(\nabla r \cdot w).$$

- L_2 is self-adjoint in second component of $\mathcal{H}$.
- At leading order only depends on $\nabla \cdot w$.

Add matching curl operator:

$$L_3 = \kappa r^{-\frac{1}{\kappa}} \operatorname{div} r^{1+\frac{1}{k}} \operatorname{curl} = \kappa \operatorname{div} r \operatorname{curl} + \nabla r \operatorname{curl}$$

- Commuting, $L_2 L_3 = L_3 L_2 = 0$.
- $L_2 + L_3$ is self-adjoint, coercive in second component of $\mathcal{H}$.

Difference bounds I

Main challenge: Compare states (r_1, v_1), (r_2, v_2) on different domains !

a) Nondegenerate distance functional $D_{\mathcal{H}}((r_1, v_1), (r_2, v_2))$:

$$D_{\mathcal{H}} = \int_\Omega (r_1 + r_2)^{\frac{1}{\kappa}-1} \left((r_1 - r_2)^2 + (r_1 + r_2)(v_1 - v_2)^2\right) dx, \quad \Omega = \Omega_1 \cap \Omega_2$$

- Good at measuring distance
- Not so good for propagation !

b) Degenerate distance functional:

$$\tilde{D}_{\mathcal{H}} := \int_\Omega (r_1 + r_2)^{\frac{1}{\kappa}-1} \left(a(r_1, r_2)(r_1 - r_2)^2 + \kappa b(r_1, r_2)(v_1 - v_2)^2\right) dx,$$

- Not so good at [directly] measuring distance
- Good for propagation !

Difference bounds II

Proposition (Equivalence)

Assume that $A = A_1 + A_2$ is small. Then

$$D_{\mathcal{H}}((r_1, v_1), (r_2, v_2)) \approx_A \tilde{D}_{\mathcal{H}}((r_1, v_1), (r_2, v_2)).$$

Proposition (Growth)

Assume that $A = A_1 + A_2$ is small. Then

$$\frac{d}{dt}\tilde{D}_{\mathcal{H}}((r_1, v_1), (r_2, v_2)) \lesssim (B_1 + B_2) D_{\mathcal{H}}((r_1, v_1), (r_2, v_2)).$$

- Proof uses delicate boundary layer analysis.

Combining the two and using Gronwall's inequality, we obtain

$$D_{\mathcal{H}}((r_1, v_1)(t), (r_2, v_2)(t)) \lesssim C(A) e^{C(A)\int_0^t B(s)ds} D_{\mathcal{H}}((r_1, v_1)(0), (r_2, v_2)(0))$$

Energy estimates I

$\mathbf{H}^{2k}$ energy E^{2k} with two components:

$$E^{2k}(r,v) = E^{2k}_{wave}(r,v) + E^{2k}_{transport}(r,v)$$

a) The wave energy is constructed using iterated material derivatives

$$(r_{2k}, v_{2k}) = (D_t^{2k} r, D_t^{2k} v)$$

These are used to construct Alihnac style good variables

$$(s_{2k}, w_{2k}) = (r_{2k} - \nabla r \cdot w_{2k-1}, v_{2k}), \qquad k \geq^{*} 0$$

viewed as multilinear expressions in (r,v) at fixed time.

$$E^{2k}_{wave}(r,v) = \sum_{j=0}^{k} \|(s_{2k}, w_{2k})\|^2_{\mathcal{H}}$$

b) The transport energy is defined in terms of the vorticity

$$E^{2k}_{transport}(r,v) = \|\omega\|^2_{H^{2k-1,k+\frac{1}{\kappa}}}$$

The energy properties

Theorem

The energy functional E^{2k} in $\mathcal{H}^{2k}$ has the following two properties:
a) Norm equivalence (coercivity):

$$E^{2k}(r,v) \approx_A \|(r,v)\|^2_{\mathcal{H}^{2k}}.$$

b) Energy estimate:

$$\frac{d}{dt}E^{2k}(r,v) \lesssim_A B\|(r,v)\|^2_{\mathcal{H}^{2k}}.$$

- Choice of energy functional is not unique, particularly for the transport component.
- By finite speed of propagation, the scale invariant parameter A can be always assumed to be small.

Energy estimates: the good variables

a) Coercivity is based on recurrence type relations

$$s_{2j} = L_1 s_{2j-2} + f_{2j}, \qquad w_{2j} = L_2 w_{2j-2} + g_{2j},$$

where f_{2j} and g_{2j} are balanced multilinear expressions of the same order, and are estimated perturbatively using interpolation inequalities.

b) Growth bounds are based on the linearized equation

$$Linearized\, Euler(s_{2k}, w_{2k}) = (F_{2k}, G_{2k})$$

where F_{2k} and G_{2k} are also balanced multilinear expressions of the same order, estimated perturbatively using interpolation inequalities.

Balanced = Products of at least two multilinear expressions with positive orders.

Existence of solutions: time discretization

Time-step $\epsilon > 0$. Given initial data $(r_0, v_0) \in \mathbf{H}^{2k}$ we produce a discrete approximate solution $(r(j\epsilon), v(j\epsilon))$, with the following properties:

a) Norm bound:

$$E^{2k}(r((j+1)\epsilon), v((j+1)\epsilon)) \leq (1 + C\epsilon)E^{2k}(r((j\epsilon), v(j\epsilon)).$$

b) Approximate solution:

$$\begin{cases} r((j+1)\epsilon) - r(j\epsilon) + \epsilon\left[v(j\epsilon)\nabla r(j\epsilon) + \kappa r(j\epsilon)\nabla \cdot v(j\epsilon)\right] = O(\epsilon^2) \\ v((j+1)\epsilon) - v(j\epsilon) + \epsilon\left[(v(j\epsilon)\cdot\nabla)v(j\epsilon) + \nabla r(j\epsilon)\right] = O(\epsilon^2) \end{cases}$$

- It suffices to do this at high regularity.
- It suffices to carry out a single iteration.
- Constant C depends only on $\mathbf{H}^{2k}$ norm of k-th iterate.

Existence of solutions: The one iteration

Idea I: Newton method [loses 2 derivatives]

$$r_1 = r_0 - \epsilon \left[v_0 \nabla r_0 + \kappa r_0 \nabla \cdot v_0 \right]$$
$$v_1 = v_0 - \epsilon \left[(v_0 \cdot \nabla) v_0 + \nabla r_0 \right]$$

Idea II: Transport + Newton method [loses 1 derivative]

$$x_1 = x_0 + \epsilon v_0(x_0)$$
$$r_1(x_1) = \left[r_0 - \epsilon \kappa r_0 \nabla \cdot v_0 \right](x_0)$$
$$v_1(x_1) = \left[v_0 - \epsilon \nabla r_0 \right](x_0)$$

Idea III: Regularization+ Transport + Newton method [no loss !]
Precede the above with a regularization step,

$$(r_0, v_0) \to (r, v)$$

Then apply the previous idea with (r_0, v_0) replaced by (r, v).

Existence: the regularization step

Proposition

Given $(r_0, v_0) \in \mathbf{H}^{2k}$ *with size* M, *there exists a regularization* (r, v) *with the following properties:*

$$i) \quad (r - r_0, v - v_0) = O(\epsilon^2), \qquad \textit{(small error)}$$

$$ii) \quad E^{2k}(r, v) \leq (1 + C(M)\epsilon) E^{2k}(r_0, v_0) \qquad \textit{(energy bound)}$$

$$iii) \quad \|(r, v)\|_{\mathcal{H}^{2k+2}} \lesssim \epsilon^{-1} M, \qquad \textit{(parabolic regularization scale)}.$$

Difficulties:

- Regularization must be adapted to acoustic metric.
- The domain has to change in the regularization.
- Regularization must also be adapted to the energy.

Rough solutions as limit of smooth solutions

Frequency envelope based approach:

i) Start with a rough data $(r_0, v_0) \in \mathcal{H}^{2k}_{r_0}$, and frequency envelope c_h.

ii) Consider the regularized initial data (r_0^h, v_0^h), with estimates

$$\|(r_0^h, v_0^h)\|_{\mathbf{H}^{2k}} \lesssim_A \|(r_0, v_0)\|_{\mathbf{H}^{2k}}. \qquad \text{(uniform bound)}$$

$$\|(r_0^h, v_0^h)\|_{\mathbf{H}_h^{2k+2j}} \lesssim 2^{2hj} c_h, \qquad j > 0. \qquad \text{(higher regularity)}$$

$$D((r_0^{h+1}, v_0^{h+1}), (r_0^h, v_0^h)) \lesssim 2^{-2hk} c_h \quad \text{(low frequency difference bound)}$$

iii) Use energy and difference bounds to propagate these estimates to the associated solutions (r^h, v^h).

iv) Prove that the limit (r, v) exists in a weaker topology, e.g. $C^1 \times \dot{C}^{\frac{1}{2}}$.

v) Show convergence in the strong topology,

$$\|(r^h, v^h) - \Psi^h(r, v)\|^2_{\mathcal{H}^{2k}} \lesssim c^2_{\geq h} := \sum_{j \geq h} c_j^2$$

Continuous dependence on data in H^{2k}

a) Start with a sequence of data $(r_{j0}, v_{j0}) \to (r_0, v_0)$ in $\mathbf{H}^{2k}$; their associated frequency envelopes c_j, c can be chosen so that $c_j \to c$ in L^2.

b) The corresponding regularized data $(r^h_{j0}, v^h_{j0}) \to (r^h_0, v^h_0)$ in all higher topologies, and so will the corresponding solutions $(r^h_j, v^h_j) \to (r^h, v^h)$.

c) The $\mathbf{H}^{2k}$ "distance" between (r^h_j, v^h_j) and (r_j, v_j) is estimated by $c_{j,\geq h} \to c_{\geq h}$ so we can make it small uniformly in j.

d) Letting $h \to \infty$, we see that the regularized solutions (r^h_j, v^h_j) converge to (r_j, v_j) uniformly.

For an overview of the use of frequency envelopes in the study of local well-posedness applied in a simpler quasilinear setting, see our recent notes on the arxiv.

Thank you !