# Angled crested like water waves

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Water Waves and Other Interface Problems Seminar
Mathematical Sciences Research Institute, Berkeley, CA
March 2, 2021

# The problem

Figure: Manhattan beach wave

# The problem (waves of interest)

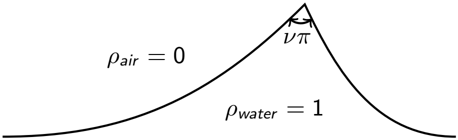



Figure: A wave with sharp crest

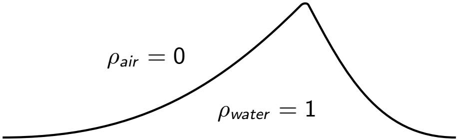
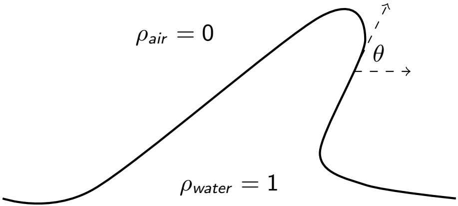



Figure: A smooth wave with large curvature/large $C^{1,\alpha}$ norm $(0 < \alpha \leq 1)$

## The problem (assumptions)




- Fluid region $\Omega(t) \subset \mathbb{R}^2$
- Zero viscosity
- Incompressible and irrotational
- Infinite depth and interface tends to flat at infinity
- Constant gravity $g = 1$ and surface tension coefficient $\sigma \geq 0$

## The problem (Euler equation)

As $\Omega(t) \subset \mathbb{R}^2 \simeq \mathbb{C}$, we let $i = \sqrt{-1}$.

$$\begin{aligned} v_t + (v \cdot \nabla)v &= -\nabla P - i && \text{in } \Omega(t) \\ \nabla \cdot v = 0 \qquad \nabla \times v &= 0 && \text{in } \Omega(t) \end{aligned}$$

where $v : \Omega(t) \to \mathbb{C}$, $P : \Omega(t) \to \mathbb{R}$.

$$\begin{aligned} &P = -\sigma \partial_s \theta \qquad \text{on } \partial\Omega(t) \\ &(1, v) \text{ is tangent to the free surface } (t, \partial\Omega(t)) \\ &v \to 0, v_t \to 0 \qquad \text{as } |(x, y)| \to \infty \end{aligned}$$

Here $\partial_s =$ arc length derivative, $\partial_s \theta =$ curvature

# Special solutions: Travelling waves

Stokes waves: periodic traveling waves, infinite depth, zero surface tension, for fixed wavelength $\lambda$ solutions parameterized by height $H$, $\lambda$ and $H$ uniquely determine speed $c$.

Stokes (1880), Toland (78), Amick-Fraenkel-Toland (82), Plotnikov (82), Plotnikov-Toland (04), Varvaruca-Weiss (11), Constantin (12)

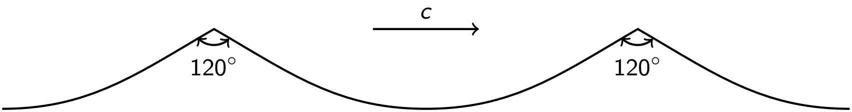
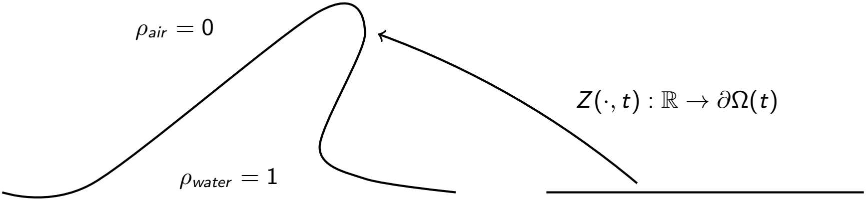



Figure: Stokes wave of greatest height

Stokes waves are unstable: Benjamin-Feir (67), Bridges-Mielke (95), Deconinck-Oliveras (11), Nguyen-Strauss (20), Hur-Yang (20), Chen-Su (20)

See also: Wilkening (11), Clamond-Henry (20)

# The Cauchy problem




- $\nabla \cdot v = 0 \qquad \nabla \times v = 0 \implies \bar{v} : \Omega(t) \to \mathbb{C}$ is holomorphic
- Take divergence to the Euler equation

$$\Delta P = -|\nabla \mathbf{v}|^2 \qquad \text{in } \Omega(t)$$
$$P = -\sigma \partial_s \theta \qquad \text{on } \partial\Omega(t)$$

- Need to solve for $\partial\Omega(t), v|_{\partial\Omega(t)}$
- Initial data in Riemann mapping coordinates is $Z(\cdot, 0), Z_t(\cdot, 0)$ where $D_t Z = Z_t$ and $D_t =$ material derivative.

# Previous works (Local wellposedness for $\sigma = 0$)

$\partial_t \sim \partial_\alpha^{1/2}$. So $Z_\alpha - 1 \in H^s(\mathbb{R}), Z_t \in H^{s+\frac{1}{2}}(\mathbb{R})$

- Small data local existence:
  Nalimov (74), Yoshihara (82), Craig (85)
- Local wellposedness:
  Wu (97,99) $s \geq 4$, Christodoulou-Lindblad (00), Lannes (05), Lindblad (05), Coutand-Shkoller (07), Zhang-Zhang (08), Castro-Córdoba-Fefferman-Gancedo-Gómez Serrano (12), Alazard-Burq-Zuily (14), Kukavica-Tuffaha (14), Hunter-Ifrim-Tataru (16), Griffiths-Ifrim-Tataru (17), Alazard-Burq-Zuily (18), Poyferré (19), Ai (19,20), Ai-Ifrim-Tataru (19) $C^{1.25}$ interfaces, Wu (20)

## Previous works (Local wellposedness for $\sigma > 0$)

$\partial_t \sim \partial_\alpha^{3/2}$. So $Z_\alpha - 1 \in H^s(\mathbb{R}), Z_t \in H^{s-\frac{1}{2}}(\mathbb{R})$

- Small data local existence:
  Yoshihara (83)
- Local wellposedness for fixed $\sigma > 0$ ($T \to 0$ as $\sigma \to 0$)
  Beyer-Gunther (98), Iguchi (01), Ambrose (03), Coutand-Shkoller (07), Christianson-Hur-Staffilani (10), Shatah-Zeng (11), Alazard-Burq-Zuily (11), Poyferré-Nguyen (16,17), Nguyen (17) $C^{2.25+}$ interfaces
- Zero surface tension limit: ($T$ uniform for $0 \leq \sigma \leq \sigma_0$)
  Ambrose-Masmoudi (05,09), Shatah-Zeng (08), Ming-Zhang (09), Castro-Córdoba-Fefferman-Gancedo-Gómez Serrano (12), Shao-Shih (18)

In both types of results $T \to 0$ as $\kappa \to \infty$ where $\kappa =$ curvature (irrespective of the value of $\sigma$)

## Previous works

- Small data long/global existence:
  Wu (09,11), Germain-Masmoudi-Shatah (12,15), Ionescu-Pusateri (15), Alazard-Delort (15), Hunter-Ifrim-Tataru (16), Ifrim-Tataru (17), Griffiths-Ifrim-Tataru (17), Wang (17), Deng-Ionescu-Pausader-Pusateri (17), Berti-Delort (18), Ionescu-Pusateri (18), Berti-Feola-Pusateri (18), Su (18), Ai-Ifrim-Tataru (19), Wang (19), Wu (20)
- Splash singularity:
  Castro-Cordoba-Fefferman-Gancedo-Serrano (13), Coutand-Shkoller (14)
- Two fluids:
  Cheng-Coutand-Shkoller (08), Shatah-Zeng (11), Lannes (13)
- Compressible fluids:
  Tanaka and Tani (03), Lindblad (05), Jang-Masmoudi (09), Coutand-Lindblad-Shkoller (10), Coutand-Shkoller (11,12), Jang-Masmoudi (15), Jang-LeFloch-Masmoudi (16), Lindblad-Luo (18), Hadžić-Shkoller-Speck (19), Disconzi-Kukavica (19), Ginsberg (19), Miao-Shahshahani-Wu (20), Ifrim-Tataru (20), Disconzi-Ifrim-Tataru (20)

## The system

- Initial data in Riemann mapping coordinates is $Z(\cdot, 0), Z_t(\cdot, 0)$ where $D_t Z = Z_t$ and $D_t =$ material derivative.
- The system is in the variables $(Z_\alpha, Z_t)$ satisfying

$$D_t Z_\alpha = Z_{t\alpha} - b_\alpha Z_\alpha$$

$$D_t \bar{Z}_t = i - i\frac{A_1}{Z_\alpha} + \frac{\sigma}{Z_\alpha}\partial_\alpha(\mathbb{I} + \mathbb{H})\left\{ \mathrm{Im}\left( \frac{1}{Z_\alpha}\partial_\alpha \frac{Z_\alpha}{|Z_\alpha|} \right) \right\}$$

where

$$b = \mathrm{Re}(\mathbb{I} - \mathbb{H})\left( \frac{Z_t}{Z_\alpha} \right)$$

$$A_1 = 1 - \mathrm{Im}[Z_t, \mathbb{H}]\bar{Z}_{t\alpha}$$

$$\mathbb{H} = \text{Hilbert transform}$$

$$= \text{ Fourier multiplier with symbol } - sgn(\xi)$$

$$D_t = \partial_t + b\partial_\alpha$$

## The Quasilinear equations for $\sigma = 0$

The quasilinear equation is

$$\left(D_t^2 + \left(-\frac{\partial P}{\partial \hat{n}}\right)\frac{1}{|Z_\alpha|}|\partial_\alpha|\right)f = l.o.t$$

For $f = \theta$ or $Z_t$.

- $|\partial_\alpha| = \sqrt{-\Delta} = i\mathbb{H}\partial_\alpha =$ Fourier multiplier with symbol $|\xi|$
- Linearize around zero solution

$$\left(\partial_t^2 + |\partial_\alpha|\right)f = 0$$

# Taylor sign condition

- The Taylor sign condition is

$$-\frac{\partial P}{\partial \hat{n}} \geq c > 0$$

See Taylor (50), Ebin (87), Beale-Hou-Lowengrub (93)

- Wu (97) proved that for $\sigma = 0$, infinite depth

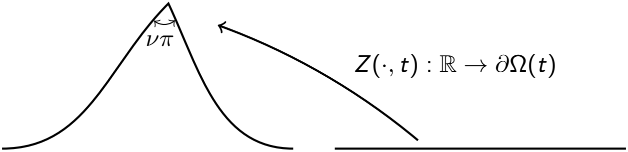
$$-\frac{\partial P}{\partial \hat{n}} = \frac{A_1}{|Z_\alpha|}$$

$A_1$ satisfies $1 \leq A_1 \leq 1 + \|Z_{t\alpha}\|_{L^2}^2$. Hence $A_1 \approx 1$.

- If the interface is $C^{1,\alpha}$ then $0 < c_1 \leq \frac{1}{|Z_\alpha|} \leq c_2 < \infty$. Hence Taylor sign condition is satisfied for $C^{1,\alpha}$ interfaces.

See also: Lannes (05), Hunter-Ifrim-Tataru (16), Su (20)

# Non $C^1$ interfaces




- If the interface has an angle of $\nu\pi$ at $\alpha = 0$ then

$$Z(\alpha) \sim \alpha^{\nu} \qquad Z_\alpha(\alpha) \sim \alpha^{\nu-1} \qquad \frac{1}{Z_\alpha}(\alpha) \sim \alpha^{1-\nu}$$

- Taylor sign condition is only satisfied in a weak sense $-\frac{\partial P}{\partial \hat{n}} = \frac{A_1}{|Z_\alpha|} \geq 0$ for $0 < \nu < 1$.
- Hence the quasilinear equation

$$\left(D_t^2 + \left(-\frac{\partial P}{\partial \hat{n}}\right)\frac{1}{|Z_\alpha|}|\partial_\alpha|\right) f = l.o.t \qquad (1)$$

around $\alpha = 0$ behaves like

$$\left(\partial_t^2 + |\alpha|^{2-2\nu}|\partial_\alpha|\right) f = l.o.t$$

# Heuristic energy estimate

Now

$$\left(\partial_t^2 + |\alpha|^{2-2\nu}|\partial_\alpha|\right) f = |\alpha|^{1-2\nu} f + \text{other l.o.t}$$

Multiply by $\partial_t f$ and integrate

$$\frac{1}{2}\frac{d}{dt}\left\{\|\partial_t f\|_{L^2}^2 + \left\||\alpha|^{1-\nu} f\right\|_{\dot{H}^{\frac{1}{2}}}\right\} \approx \int (\partial_t f)\left(|\alpha|^{1-2\nu} f\right) d\alpha + \cdots$$

- We can harmlessly add $\|f\|_2^2$ to the energy and is compatible with the energy.
- As $f \in L^2$ and we want $|\alpha|^{1-2\nu} f \in L^2$, we need $\nu \leq \frac{1}{2}$.

Note:

- Smaller angles are better than bigger angles with $\pi/2$ being the threshold.
- Harmonic functions have better regularity in corners of smaller angles.
- This threshold of $\pi/2$ also shows up in the uniqueness of Yudovich solutions for the 2D Euler equation on corner domains. (See Agrawal-Nahmod (2020))

# Local wellposedness for $\sigma = 0$

Kinsey and Wu (14) - A priori estimates, Wu (18) - Existence and uniqueness

- Allows angled crests as initial data with angles $\nu\pi$ with $0 < \nu < \frac{1}{2}$.
- Weighted $H^s$ norm and interfaces are $C^{2.5}$ a.e. Weights are powers of $\frac{1}{|Z_\alpha|} \approx |\alpha|^{1-\nu}$

Agrawal (19) lowered the regularity of the energy of Kinsey and Wu (14) to the interface being $C^2$ a.e.

$$\mathcal{E}(t) = \left\| \partial_\alpha \frac{1}{Z_\alpha} \right\|_{L^2}^2 + \left\| \frac{1}{Z_\alpha} \partial_\alpha \frac{1}{Z_\alpha} \right\|_{\dot{H}^{\frac{1}{2}}}^2 + \left\| \bar{Z}_{t\alpha} \right\|_{L^2}^2 + \left\| \frac{1}{Z_\alpha^2} \partial_\alpha \bar{Z}_{t\alpha} \right\|_{L^2}^2$$

Questions left open from Kinsey and Wu (14), Wu (18):

- Are there other singularities allowed by the energy?
- How does the angle change with time? What are the dynamics of the singularities?
- What happens to the particle at the corner?

In Kinsey and Wu (14), a heuristic argument given to show that the angles do not change

## Main result 1 (Rigidity of singularities, $\sigma = 0$)

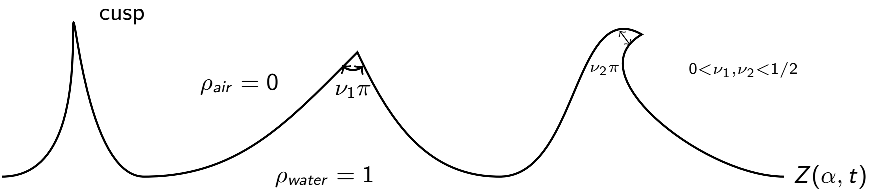



Figure: A wave with angled crests and cusps

### Theorem (Agrawal 18)

*The existence result of Wu (18) allows interfaces with cusps. Moreover as long as the energy remains finite we have*

- *Interface with angled crests/cusps remain angled crested/cusped*
- *Angles do not change nor tilt*
- *Particles at the tip stay at the tip*
- $v, v_t, \nabla v, \nabla P$ *extend continuously to the boundary and the Euler equation holds even on the boundary*
- $\nabla v = \nabla P = 0$ *at the tip. Hence acceleration at the tip* $= -i$

## Quasilinear equations for $\sigma > 0$

- A computation shows that (proved in Agrawal 19)

$$-\frac{\partial P}{\partial \hat{n}} = \frac{1}{|Z_\alpha|}(A_1 + \sigma|\partial_\alpha|\kappa)$$

  where $\kappa =$ curvature, $A_1 \geq 1$ is lower order.
- Hence Taylor sign condition fails generically if $\sigma$ is large.
- The general quasilinear equation (derived in Agrawal 19) is

$$\left(D_t^2 + \left(-\frac{\partial P}{\partial \hat{n}}\bigg|_{\sigma=0}\right)\frac{1}{|Z_\alpha|}|\partial_\alpha| - \sigma\left(\frac{1}{|Z_\alpha|}\partial_\alpha\right)^2\frac{1}{|Z_\alpha|}|\partial_\alpha|\right) f = l.o.t$$

  for $f = \frac{1}{|Z_\alpha|}\partial_\alpha\theta$ or $D_t\theta$.
- Note that $-\frac{\partial P}{\partial \hat{n}}\Big|_{\sigma=0} = \frac{A_1}{|Z_\alpha|} \geq 0$
- Linearize around zero solution

$$\left(\partial_t^2 + |\partial_\alpha| + \sigma|\partial_\alpha|^3\right) f = 0$$

# Main result 2 (Existence, $\sigma > 0$)

Define

$$\mathcal{E}_{\sigma,1} = \left\|\partial_\alpha \frac{1}{Z_\alpha}\right\|_{L^2}^2 + \left\|\frac{1}{Z_\alpha}\partial_\alpha \frac{1}{Z_\alpha}\right\|_{\dot{H}^{\frac{1}{2}}}^2 + \left\|\frac{\sigma^{\frac{1}{2}}}{Z_\alpha^{\frac{1}{2}}}\partial_\alpha^2 \frac{1}{Z_\alpha}\right\|_2^2 + \cdots$$

$$\mathcal{E}_{\sigma,2} = \left\|\bar{Z}_{t\alpha}\right\|_{L^2}^2 + \left\|\frac{1}{Z_\alpha^2}\partial_\alpha \bar{Z}_{t\alpha}\right\|_{L^2}^2 + \left\|\frac{\sigma^{\frac{1}{2}}}{Z_\alpha^{\frac{1}{2}}}\partial_\alpha \bar{Z}_{t\alpha}\right\|_{L^2}^2 + \cdots$$

$$\mathcal{E}_\sigma = \mathcal{E}_{\sigma,1} + \mathcal{E}_{\sigma,2}$$

## Theorem (Agrawal 19)

*Let $\sigma > 0$ and assume that $\mathcal{E}_\sigma(0) < \infty$ and $Z_\alpha(\cdot,0) - 1, \bar{Z}_t(\cdot,0) \in L^2$. Then there are constants $T = T(\mathcal{E}_\sigma(0)) > 0$ and $C = C(\mathcal{E}_\sigma(0)) > 0$ depending only on $\mathcal{E}_\sigma(0)$ and a unique solution $(Z(\cdot,t), Z_t(\cdot,t))$ to the capillary gravity water wave equation in $[0,T]$ so that*

$$\sup_{[0,T]} \mathcal{E}_\sigma(t) \le C(\mathcal{E}_\sigma(0)) < \infty$$

## Main result 2 ($\sigma > 0$)

**Properties:**

- Energy is positive for all $\sigma$: No assumptions on the Taylor sign condition. Also $\mathcal{E}_\sigma$ is an increasing function of $\sigma$.
- If we fix an initial data $(Z_\alpha - 1, Z_t) \in H^{s+1/2} \times H^s$ with $s \geq 3$, then for arbitrary $\sigma_0 > 0$ we have a uniform time of existence $T_0$ (depending only on $\sigma_0$) for all $0 \leq \sigma \leq \sigma_0$.
- Energy allows angled crest solutions for $\sigma = 0$. Also in this case, energy is lower order by half spacial derivatives as compared to the energy of Kinsey and Wu
- Energy does not allow angled crest solutions for $\sigma > 0$: If $\sigma > 0$ and $\mathcal{E}_\sigma < \infty$ then the interface is $C^4$. However we get the estimate $\|\kappa\|_{L^\infty} \leq \sigma^{-\frac{1}{3}} C(\mathcal{E}_\sigma)$ where $\kappa$ is the curvature. Hence energy allows interface with large curvature.

# Application

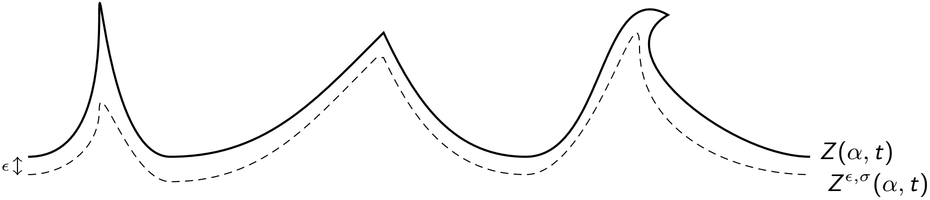



Figure: Waves with and without surface tension

**Corollary (Agrawal 19, Agrawal 20))**

- *Let* $0 < \epsilon \leq 1$ *and* $\dfrac{\sigma}{\epsilon^{3/2}} \leq 1$*, then there exists* $T > 0$ *independent of* $\epsilon, \sigma$ *so that the solutions* $(Z^{\epsilon,\sigma}, Z_t^{\epsilon,\sigma})$ *exist in* $[0, T]$
- *If in addition* $\epsilon, \sigma \to 0$ *with* $\dfrac{\sigma}{\epsilon^{3/2}} \to 0$*, then* $(Z^{\epsilon,\sigma}, Z_t^{\epsilon,\sigma}) \to (Z, Z_t)$ *in* $[0, T]$ *with* $\mathcal{E}_\Delta(Z^\epsilon, Z^{\epsilon,\sigma}) + \mathcal{F}(Z, Z^\epsilon) \to 0$.

- Heuristically this says that if $\sigma \lesssim \epsilon^{\frac{3}{2}}$ then the interface does not feel the effect of surface tension for $O(1)$ time.
- If we put $\sigma = \epsilon^{\frac{3}{2}}$ and $\nu = \frac{1}{2} - \frac{3}{2}\delta$ we obtain $\|\kappa^{\epsilon,\sigma}\|_{L^\infty}(0) \sim \sigma^{-\frac{1}{3}+\delta}$ as $\sigma \to 0$.

# Heuristic energy estimate

The quasilinear equation is

$$\left(D_t^2 + \left(\frac{A_1}{|Z_\alpha|}\right)\frac{1}{|Z_\alpha|}|\partial_\alpha| - \sigma\left(\frac{1}{|Z_\alpha|}\partial_\alpha\right)^2\frac{1}{|Z_\alpha|}|\partial_\alpha|\right)f = l.o.t \qquad (2)$$

If the interface has an angled crest of angle $\nu\pi$ at $\alpha = 0$, then $Z(\alpha) \sim \alpha^\nu$ and hence $\frac{1}{|Z_\alpha|} \sim |\alpha|^{1-\nu}$ near $\alpha = 0$ and hence the quasilinear equation near $\alpha = 0$ behaves like

$$\left\{\partial_t^2 + |\alpha|^{2-2\nu}|\partial_\alpha| + \sigma|\alpha|^{3-3\nu}|\partial_\alpha|^3\right\}f$$
$$= |\alpha|^{1-2\nu}f + \sigma|\alpha|^{2-3\nu}|\partial_\alpha|^2 f + \sigma|\alpha|^{1-3\nu}|\partial_\alpha|f + \sigma|\alpha|^{-3\nu}f + \text{other l.o.t}$$

Multiply by $\partial_t f$ and integrate

$$\frac{1}{2}\frac{d}{dt}\left\{\|\partial_t f\|_{L^2}^2 + \left\||\alpha|^{1-\nu}f\right\|_{\dot{H}^{\frac{1}{2}}} + \left\|\sigma^{\frac{1}{2}}|\alpha|^{\frac{3}{2}-\frac{3}{2}\nu}|\partial_\alpha|f\right\|_{\dot{H}^{\frac{1}{2}}}\right\}$$
$$\approx \int(\partial_t f)\left(|\alpha|^{1-2\nu}f + \sigma|\alpha|^{2-3\nu}|\partial_\alpha|^2 f + \sigma|\alpha|^{1-3\nu}|\partial_\alpha|f + \sigma|\alpha|^{-3\nu}f\right)d\alpha$$

## Heuristic energy estimate

$$\frac{1}{2}\frac{d}{dt}\Big\{\|\partial_t f\|_{L^2}^2 + \Big\||\alpha|^{1-\nu} f\Big\|_{\dot{H}^{\frac{1}{2}}} + \Big\|\sigma^{\frac{1}{2}}|\alpha|^{\frac{3}{2}-\frac{3}{2}\nu}|\partial_\alpha| f\Big\|_{\dot{H}^{\frac{1}{2}}}\Big\}$$
$$\approx \int (\partial_t f)\Big(|\alpha|^{1-2\nu} f + \sigma|\alpha|^{1-3\nu}|\partial_\alpha| f + \sigma|\alpha|^{-3\nu} f\Big)\, d\alpha$$

- As we only have $f \in L^2$, there is no way we can control the term $\sigma|\alpha|^{-3\nu} f \in L^2$ and this is the reason why we do not allow angled crest data if $\sigma > 0$.
- If we work with the smooth interface $Z^\epsilon = Z * P_\epsilon$ where $P_\epsilon$ is the Poisson kernel, then this has the effect of changing $|\alpha| \mapsto |-i\epsilon + \alpha|$ near $\alpha = 0$. Hence to close the energy estimate, we obtain the restriction $\sigma\epsilon^{-3\nu} \lesssim 1$. Letting $\nu \uparrow \frac{1}{2}$, we get $\sigma\epsilon^{-\frac{3}{2}} \lesssim 1$
- A similar argument for $\sigma|\alpha|^{1-3\nu}|\partial_\alpha| f \in L^2$ also yields the same restriction.

## The scaling

- If $g = 0$ then for $\lambda > 0$ and $s \in \mathbb{R}$, $Z_\lambda(\alpha, t) = \lambda^{-1} Z(\lambda\alpha, \lambda^s t)$ with $\sigma_\lambda = \lambda^{2s-3}\sigma$ is another solution
- We are interested in the zero surface tension limit, so we want the solutions $Z_\lambda(\cdot, t)$ to exist in the same time interval $[0, T]$. So put $s = 0$.
- Hence $Z_\lambda(\alpha, t) = \lambda^{-1} Z(\lambda\alpha, t)$ and surface tension $\sigma_\lambda = \lambda^{-3}\sigma$.
- Hence $\|\sigma^{\frac{1}{3}}\kappa\|_{L^\infty}$ is invariant under this scaling and so the curvature grows like $\sigma^{-\frac{1}{3}}$ as $\sigma \to 0$.

Thank You!