Traveling Waves with Multi-Valued Height

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Traveling waves

- There is a long, rich history of theory of traveling waves in free-surface fluid dynamics.
- Typically, with x being the horizontal spatial coordinate, the height of the free surface is $\eta(x-ct)$. Another quantity such as the potential on the free surface is also taken of the form $\phi(x-ct)$.
- The focus on functions of $x-ct$ implies a restriction to waves with single-valued height.
- There are contexts where single-valued height is natural, such as pure gravity water waves. But for more general waves, such as capillary-gravity waves, the assumption of single-valued height is artificial.
- We have two goals: (1) prove the existence of traveling waves with multi-valued height, (2) allow for the existence of multi-valued height in general existence theories of traveling waves.

Crapper waves

- Why do we want to prove the existence of traveling waves with multi-valued height? Because we already know that some exist.
- The Crapper waves are a family of exact traveling capillary water waves on infinite depth, formulas for which can be written down:

$$q \geq 1, \qquad q = \frac{1 + A^2}{1 - A^2}$$

$$\omega(z) = 2i \log\left(\frac{1 + Az}{1 - Az}\right), \qquad \theta_q(a) = \mathrm{Re}\left(\omega(e^{ia})\right).$$

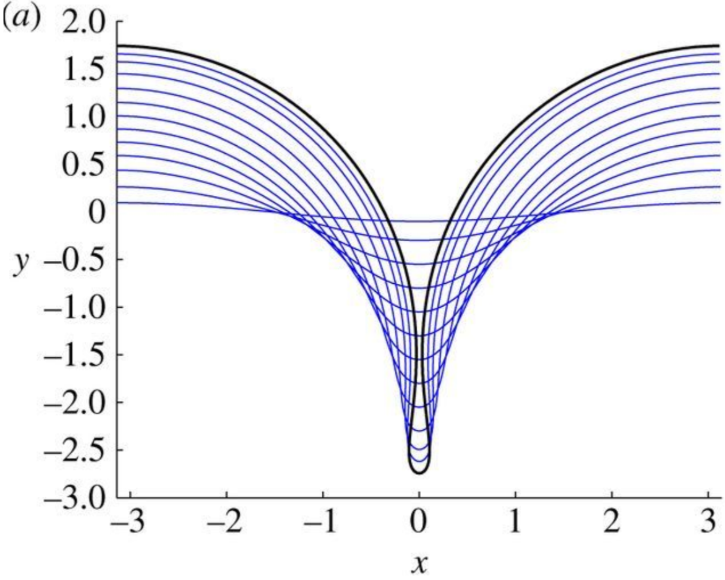
- These satisfy the traveling wave equation (see Okamoto & Shōji)

$$F(\theta; p, q) = e^{2H\theta}\frac{dH\theta}{da} - pe^{-H\theta}\sin(\theta) + q\frac{d}{da}\left(e^{H\theta}\frac{d\theta}{da}\right) = 0,$$

when the gravity parameter satisfies $p = 0$.

- Kinnersley extended these to the case of finite depth.

Crapper waves



Gravity-perturbed Crapper waves

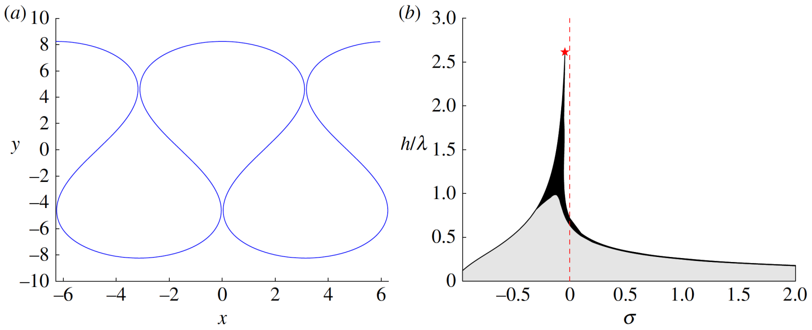
- We want to show that overturning waves exist more generally than for pure capillary water waves.
- We do so by the implicit function theorem: we have an equation for which we know solutions when $p = 0$, and we want to know solutions exist for $p \neq 0$.
- For the implicit function theorem, we mainly need to know about the linearization, Γ, of the mapping F (linearized about a Crapper wave θ_q).
- Fortunately, the book of Okamoto and Shōji has a wealth of information about the linearized operator.
- Bottom line: we need Γ to be a bijection between certain spaces, but it is not. Instead Γ is injective but not surjective. For maps which are not injective, a Lyapunov-Schmidt decomposition is frequently employed. We make the surjective analog of the Lyapunov-Schmidt decomposition.

Gravity-perturbed Crapper waves: results

Theorem 2.4. *For all $q > 1$, there exist $P = P(q) > 0$ and a C^∞ function*

$$\Theta_q : (-P, P) \longrightarrow X^2$$

such that $F(\Theta_q(p); p, q) = 0$ for $|p| < P$ and $\Theta_q(0) = \theta_q$. Moreover, $\Theta_q(p)$ is an odd function of a, and Θ_q is smooth with respect to q.



Extensions by more than gravity

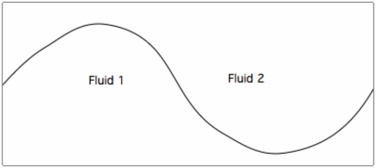
- Other authors subsequently perturbed Crapper waves by including other effects.
- de Boeck '14 included nonzero constant vorticity.
- Cordoba, Enciso, and Grubic '16, '21 include nonzero density of the upper fluid.
- Nonconstant vorticity seems likely to generate such waves as well.

Allowing multi-valued height in a traveling wave formulation

- When studying traveling waves, we would like to formulate the problem so as to allow multi-valued height, rather than introduce an artificial restriction.
- Some authors have done this in 2D flows by using a conformal map, but we would like an approach which extends simply to the 3D case.
- We will now detail such an approach which does not restrict to single-valued height and does not use complex analysis, and which is amenable to analysis and computing.
- The analysis we will describe here is a global bifurcation theorem.

Interfacial Flow

- We consider two infinitely deep, horizontally periodic fluids, separated by a sharp interface.



- The fluid velocities are given by the irrotational, incompressible Euler equations:

$$\rho_i \left(\mathbf{v}_{i,t} + \mathbf{v}_i \cdot \nabla \mathbf{v}_i \right) = -\nabla p_i,$$

$$\mathrm{div}(\mathbf{v}_i) = 0,$$

$$\mathbf{v}_i = \nabla \phi_i.$$

- The fluids have densities ρ_1 and ρ_2, each of which is constant.

Surface Tension

- Surface tension enters the problem through the Laplace-Young jump condition for the pressure: $[p] = \tau\kappa$, where τ is the surface tension coefficient and κ is the curvature of the interface.
- The initial value problem with surface tension is known to be well-posed (A '03).
- The initial value problem without surface tension is known to be ill-posed (Caflisch-Orellana, Lebeau/Kamotski, Wu), except in the water wave case ($\rho_2 = 0$).
- Hou, Lowengrub, and Shelley have introduced a numerical method for the efficient numerical solution of the initial value problem for the vortex sheet with surface tension. This method begins with a formulation of the problem using geometric variables related to the curvature.

The HLS Formulation

- The interface is $(x(\alpha, t), y(\alpha, t))$, with $(x, y)_t = U\hat{\mathbf{n}} + V\hat{\mathbf{t}}$.
- Hou, Lowengrub, and Shelley describe the curve with its tangent angle, θ, and arclength element, s_α.
- Then, $s_{\alpha,t} = V_\alpha - \theta_\alpha U$.
- If $L(t)$ is the length of one period, then a normalized arclength parameterization sets $s_\alpha = L/2\pi$.
- This defines V :
$$V_\alpha = \theta_\alpha U + \frac{L_t}{2\pi}.$$
- With these choices, the curvature has become essentially linear:
$$\kappa = \left(\frac{\theta}{s_\alpha}\right)_\alpha = \frac{2\pi}{L}\theta_\alpha.$$

The Normal Velocity

- The normal velocity is dictated by the fluid dynamics. This implies that $U = \mathbf{W} \cdot \hat{\mathbf{n}}$, where $\mathbf{W}$ is the Birkhoff-Rott integral.
- Using complex notation, denote the interface as $z = x + iy$. The Birkhoff-Rott integral is $\mathbf{W} = (W_1, W_2)$, given by

$$(W_1 - iW_2)(\alpha) = \frac{1}{4\pi i}\mathrm{PV}\int \gamma(\alpha')\cot\left(\frac{1}{2}(z(\alpha) - z(\alpha'))\right)\, \mathrm{d}\alpha'.$$

- Here, γ is the vortex sheet strength; this is proportional to the jump in the tangential velocity across the interface.
- We have an evolution equation for γ :

$$\gamma_t = \tau\frac{\theta_{\alpha\alpha}}{s_\alpha} + \frac{((V - \mathbf{W}\cdot\hat{\mathbf{t}})\gamma)_\alpha}{s_\alpha}.$$

- This γ_t equation is in the density matched case; in the more general case, there are several additional terms proportional to $\rho_1 - \rho_2$.

A Traveling Parameterized Curve

- The usual traveling wave formulation is to posit that the dependent variables are functions of $x - ct$, for some wave speed c.
- This is fine for us at small amplitudes, but we should be able to find large-amplitude solutions as well.
- At large amplitudes, our traveling vortex sheets with surface tension should be able to overturn, i.e., we would like to allow for waves with multi-valued height.
- Instead of studying functions of $x - ct$, we must make a traveling wave ansatz for a parameterized curve.
- With α again corresponding to the normalized arclength parameterization, we seek solutions for which

$$(x(\alpha, t), y(\alpha, t))_t = (c, 0).$$

The Traveling Wave Ansatz

- We have two expressions for the velocity of the curve:

$$(x, y)_t = (c, 0),$$

$$(x, y)_t = U\hat{\mathbf{n}} + V\hat{\mathbf{t}}.$$

- We also have an expression for the tangential velocity:

$$V_\alpha = \mathbb{P}(\theta_\alpha U).$$

- The solution of these equations is

$$U = -c\sin(\theta), \qquad V = c\cos(\theta).$$

- The equation for V is redundant, from the equation for V_α above.
- We still need another equation.

Completing the Traveling Wave Ansatz

- Since we have a system of evolution equations (for θ and γ), we need a pair of equations in our traveling wave ansatz.
- In addition to specifying $U = -c\sin(\theta)$, we also specify $U_t = (-c\sin(\theta))_t$.
- It's easy to see that $\theta_t = 0$, so this becomes $U_t = 0$.
- Recall that U is given by

$$U = \mathrm{Re}\left\{\frac{e^{i\theta}L}{2}\mathrm{PV}\int_0^{2\pi}\gamma(\alpha')\cot\left(\frac{1}{2}\left(z(\alpha)-z(\alpha')\right)\right)\,\mathrm{d}\alpha'\right\}.$$

- Given the equation $U = -c\sin(\theta)$, this becomes, upon differentiating with respect to t,

$$U_t = \mathrm{Re}\left\{\frac{e^{i\theta}L}{2}\mathrm{PV}\int_0^{2\pi}\gamma_t(\alpha')\cot\left(\frac{1}{2}(z(\alpha)-z(\alpha'))\right)\,\mathrm{d}\alpha'\right\}.$$

The Final Traveling Wave Ansatz

- From the above formula for U_t, we can see that if $\gamma_t = 0$, then $U_t = 0$. (We can prove the other direction as well: if $U_t = 0$, then $\gamma_t = 0$.)
- We solve the equations $U = -c\sin(\theta)$ and $\gamma_t = 0$, while using the normalized arclength parameterization.
- More explicitly, these equations are:

$$\mathrm{Re}\left\{\frac{e^{i\theta}L}{2}\mathrm{PV}\int_0^{2\pi}\gamma(\alpha')\cot\left(\frac{1}{2}\left(z(\alpha)-z(\alpha')\right)\right)\,\mathrm{d}\alpha'\right\} = -c\sin(\theta),$$

$$\frac{\tau\theta_{\alpha\alpha}}{s_\alpha} + \left(\frac{(V-\mathbf{W}\cdot\hat{\mathbf{t}})\gamma}{s_\alpha}\right)_\alpha = 0.$$

- The quantities L, V, and z are all determined by θ, and $\mathbf{W}$ is determined by θ and γ.

Bifurcation Theory

- We use an "identity plus compact" global bifurcation theorem.
- The system can be rewritten as $(\theta, \gamma) + K[\theta, \gamma] = 0$, for a compact operator K.
- Two main steps in accomplishing this formulation:
  - In the $U = -c\sin(\theta)$ equation, write U as the Hilbert transform of γ plus a smooth remainder.
  - In the $\gamma_t = 0$ equation, invert the derivatives on θ. This could be described as "inverting the curvature" to gain regularity.
- After making the "identity plus compact" formulation, some spectral information must be computed (must have odd crossing number).

Our global bifurcation result

Theorem

(Main Theorem) *For all choices of the constants $\tau > 0$, $M > 0$, $\overline{\gamma} \in \mathbb{R}$, $\rho_1, \rho_2 \geq 0$ (not both zero) and $g \in \mathbb{R}$, there exist a countable number of connected sets of smooth non-trivial symmetric periodic traveling wave solutions, bifurcating from a quiescent equilibrium, for the two-dimensional gravity-capillary vortex sheet problem. If $\bar{\gamma} \neq 0$ or $\rho_1 \neq \rho_2$, then each of these connected sets has at least one of the following properties:*

- (a) *it contains waves whose interfaces have lengths per period which are arbitrarily long;*
- (b) *it contains waves whose interfaces have arbitrarily large curvature;*
- (c) *it contains waves where the jump of the tangential component of the fluid velocity across the interface or its derivative is arbitrarily large;*

[**To be continued on next slide**]

Theorem

(Main Theorem) (continued)

(d) *its closure contains a wave whose interface has a point of self intersection;*

(e) *it contains a sequence of waves whose interfaces converge to a flat configuration but whose speeds contain at least two convergent subsequences whose limits differ.*

In the case that $\bar{\gamma} = 0$ *and* $\rho_1 = \rho_2$*, then each connected set has at least one of the properties (a)-(f), where (f) is the following:*

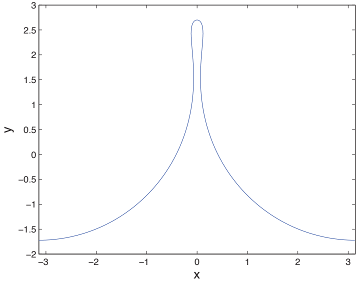
(f) *it contains waves which have speeds which are arbitrarily large.*

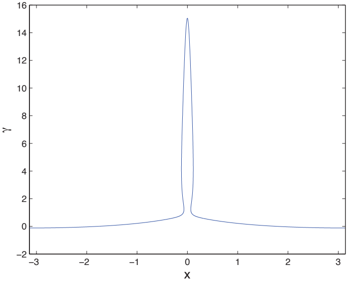
About the main theorem

- We have found six possible different ways that the bifurcation curve could end.
- Typically when one proves such a result, one then attempts to eliminate several outcomes. Eliminating an outcome could mean proving that it does not occur, or it could also mean proving that if it does occur, then one of the other items on the list must also occur, making it redundant.
- We have not been able to eliminate outcomes, and in fact, numerical evidence indicates that each of outcomes (a)-(e) do occur for some values of the parameters.
- We have not observed outcome (f) numerically, but also have not been able to eliminate it.

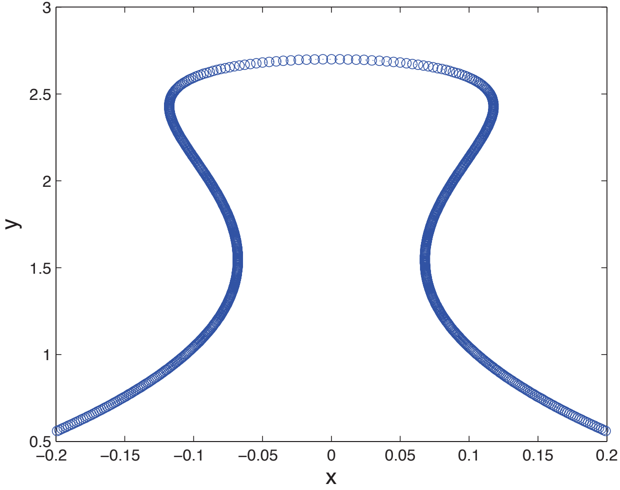
Computed Solutions

- For nonzero $\bar{\gamma}$, we find that the waves always overturn, and the bifurcation curve ends in self-intersection of the interface.



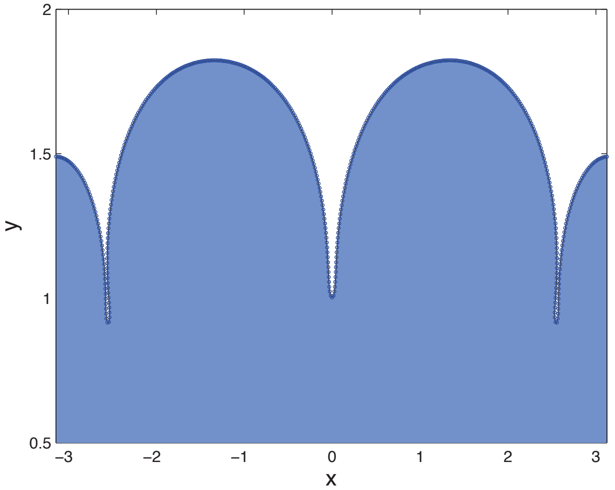


A Close-up of the Interface

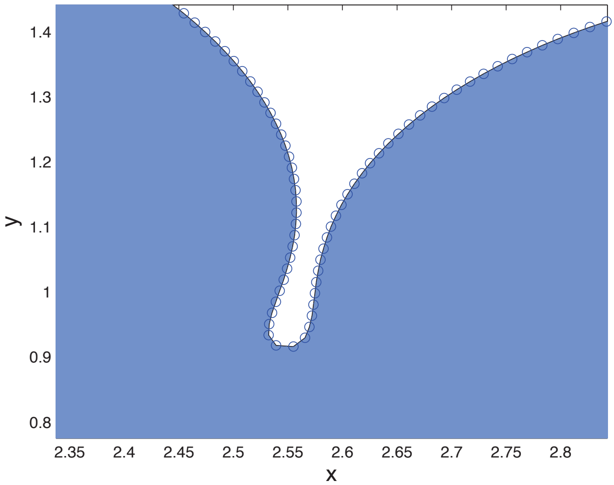


Another Traveling Wave

- A large-amplitude gravity-capillary water wave:

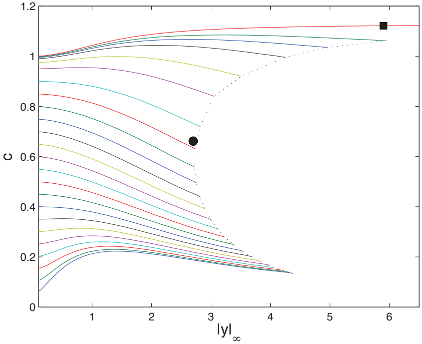


A Close-up of the Interface



About outcome (f)

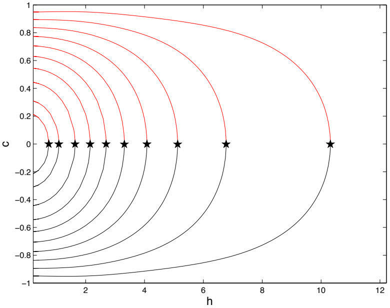
- This is the case $\rho_1 = \rho_2$
- Numerically, we fix τ and use various values of $\bar{\gamma}$.
- The $\bar{\gamma} = 0$ case is the uppermost curve in the following diagram:

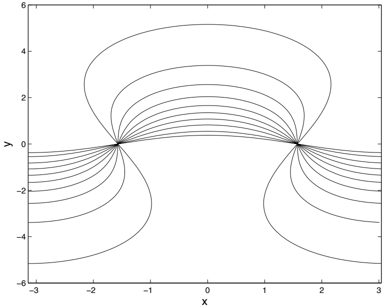


- The $\bar{\gamma} = 0$ curve is unbounded, but speed is bounded.

About outcome (e)

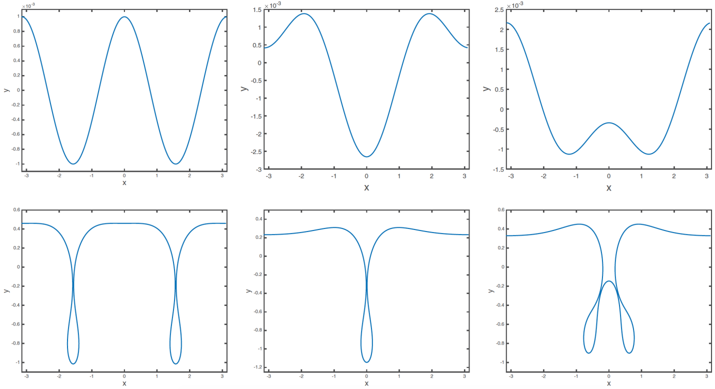
- This is the “reconnection to trivial” outcome.
- For pure gravity water waves, this outcome can be eliminated. Such an argument uses the maximum principle.
- With the larger number of derivatives stemming from the presence of surface tension, the maximum principle argument does not work.
- For certain parameter values (but always with *negative gravity*), we are able to observe this behavior.





Hydroelastic waves

- Hydroelastic waves model free surface flows with elastic effects, such as ice sheets on the ocean. We use the model of Toland and Plotnikov, and can repeat our global bifurcation theorem.
- For the crossing number, we now need to consider quartic polynomials rather than quadratic. We are able to repeat the global bifurcation theory.



Extension to 3D

- Our “traveling parameterized curve” philosophy does extend to 3D.
- Akers and Reeger computed 3D traveling waves using a simpler velocity than the Birkhoff-Rott integral; in the 3D case, computing the Birkhoff-Rott integral is computationally intensive (cf. Ambrose-Siegel-Tlupova), so they used an approximation with Riesz transforms.
- Work on analysis for the 3D problem (with the full velocity) is ongoing (with Miles Wheeler).

Thanks for your attention.