The relativistic Euler equations with a physical vacuum boundary

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Joint work M. Ifrim and D. Tataru.

Water Wave / Interface Problems

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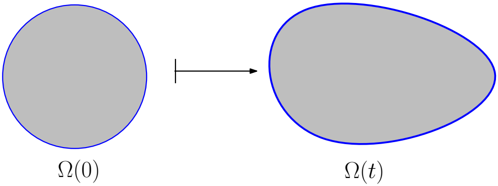

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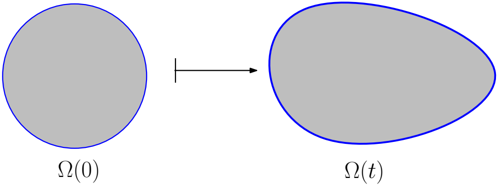

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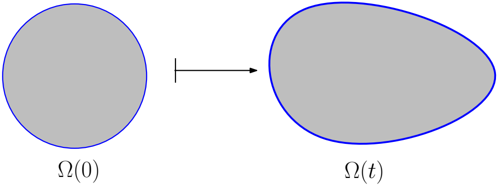

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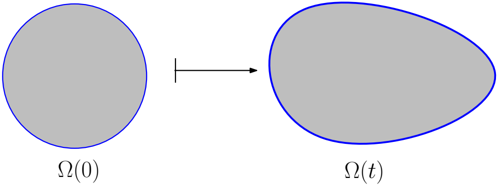

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The study of free-boundary fluids (incompresible, compressible, inviscid, viscous, classical, relativistic,...) is a very active field of research (several authors).

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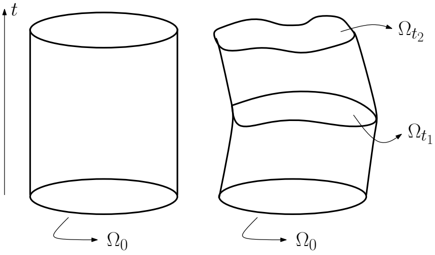
known as the **moving domain.**



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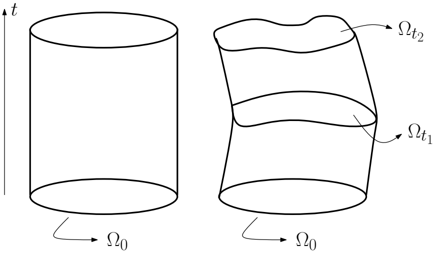

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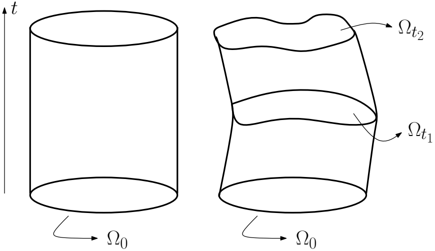

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Understanding the dynamics of Γ is essential in free-boundary problems.

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$$u^\mu \partial_\mu \varrho + (p+\varrho)\partial_\mu u^\mu = 0, \tag{1a}$$

$$(p+\varrho)u^\mu \partial_\mu u^\alpha + \Pi^{\alpha\mu}\partial_\mu p = 0, \tag{1b}$$

$$g_{\alpha\beta}u^\alpha u^\beta = -1, \tag{1c}$$

where: $\partial_\mu := \frac{\partial}{\partial x^\mu}$; $\{x^\mu\}_{\mu=0}^3$ are coordinates in spacetime, $t := x^0$, $x = (x^1, x^2, x^3)$; Greek indices $= 0, \ldots, 3$, Latin indices $= 1, \ldots, 3$, sum convention; $\varrho = \varrho(t, x)$ is the fluid's (energy) density; $u = u(t, x)$ is the fluid's (four-)velocity; $p = p(\varrho)$ is the fluid's pressure (equation of state); g is the Minkowski metric; Π is the projection onto the orthogonal (w.r.t. g) to u.

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(1c): constraint propagated by the flow.

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(Kinematic boundary condition: u is tangent to Γ.)

Physical vacuum boundary

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- A faster decay would cause the boundary to simply move independently and linearly with the outer particle's speed.
- A slower decay rate would lead to a very singular problem, likely causing an infinite initial acceleration of the boundary.

Features of the physical vacuum boundary problem

The physical vacuum boundary condition $c_s^2(t,x) \approx \mathrm{dist}(x, \Gamma_t)$ implies that linear waves with speed c_s reach the boundary in finite time. Thus, the motion of the boundary is strongly coupled to the wave evolution and is *not* just a self-contained evolution at leading order.

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A standard choice of equation of state for a physical vacuum is

$$p(\varrho) = \varrho^{\kappa+1},\ \kappa > 0,$$

which we henceforth assume.

General strategy

The key steps for tackling the problem are:

- Good nonlinear variables: diagonalize the system with respect to the material derivative $D_t := \partial_t + \frac{u^i}{u^0}\partial_i$ (choice of spacetime foliation); tailored to the characteristics.

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- Regularization $+$ time discretization to obtain solutions.

Diagonalizing the system

The rescaled velocity $v := f(\varrho)u$ satisfies

$$\frac{p+\varrho}{f} u^\mu \partial_\mu v^i + c_s^2 g^{i\mu}\partial_\mu \varrho + \left(-\frac{f'}{f}(p+\varrho) + c_s^2\right) u^i u^\mu \partial_\mu \varrho = 0$$

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This change of variables also diagonalizes the ϱ-equation:

$$v^\mu\partial_\mu\varrho + (p+\varrho)\left(\delta^{ij} - \frac{v^i v^j}{(v^0)^2}\right)\partial_i v_j - c_s^2 f^2 v^i \partial_i \varrho = 0.$$

(Omitting $O(1)$ coefficients throughout.)

The good variables

For $p(\varrho) = \varrho^{\kappa+1}$, $v := (1+\varrho^\kappa)^{1+\frac{1}{\kappa}} u$, $r := \frac{\kappa+1}{\kappa}\varrho^\kappa$ (sound speed^2),

$$D_t r + r G^{ij}\partial_i v_j + r v^i \partial_i r = 0, \tag{2a}$$

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The term $r v^i \partial_i r$ in (2a) can roughly be viewed as a perturbation; energy estimates: multiply by s and integrate

$$\int_{\Omega_t} s r v^i \partial_i s \, dx = \frac{1}{2}\int_{\Omega_t} r v^i \partial_i s^2 \, dx \lesssim \|s\|^2_{L^2(\Omega_t)}.$$

Function spaces

The linearized equations admit the following energy ($r \rightsquigarrow s, v \rightsquigarrow w$):

$$\|(s,w)\|^2_{\mathcal{H}} := \int_{\Omega_t} r^{\frac{1-\kappa}{\kappa}} (s^2 + rG^{ij} w_j w_i)\, dx.$$

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$$\begin{aligned}\|(s,w)\|_{\mathcal{H}^{2k}}^2 &:= \sum_{|\alpha|=0}^{2k} \sum_{\substack{a=0 \\ |\alpha|-a\le k}}^{k} \|r^{\frac{1-\kappa}{2\kappa}+a} \partial^\alpha s\|_{L^2(\Omega_t)}^2 \\ &\quad + \sum_{|\alpha|=0}^{2k} \sum_{\substack{a=0 \\ |\alpha|-a\le k}}^{k} \|r^{\frac{1-\kappa}{2\kappa}+\frac{1}{2}+a} \partial^\alpha w\|_{L^2(\Omega_t)}^2 \\ &\sim \|r^{\frac{1-\kappa}{2\kappa}+k} \partial^{2k} s\|_{L^2(\Omega_t)} + \|r^{\frac{1-\kappa}{2\kappa}+\frac{1}{2}+k} \partial^{2k} w\|_{L^2(\Omega_t)}.\end{aligned}$$

Scaling analysis

Ignoring $O(1)$ terms, equations

$$D_t r + rG^{ij}\partial_i v_j + rv^i\partial_i r = 0,$$
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From this we determined the critical space $\mathcal{H}^{2k_0}$ where

$$2k_0 = 3 + 1 + \frac{1}{\kappa}.$$

Theorem: LWP

Define the phase space

$$\mathbf{H}^{2k} := \left\{ (r, v) \,|\, (r, v) \in \mathcal{H}^{2k} \right\}.$$

Equations

$$\begin{aligned} D_t r + r G^{ij} \partial_i v_j + r v^i \partial_i r = 0, \\ D_t v_i + \partial_i r = 0, \end{aligned}$$

are locally well-posed in $\mathbf{H}^{2k}$ for data $(\mathring{r}, \mathring{v}) \in \mathbf{H}^{2k}$ provided that

$$\mathring{r}(x) \approx \mathrm{dist}(x, \partial\Omega_0), \quad \Omega_0 = \{\mathring{r} > 0\},$$

and

$$2k > 2k_0 + 1, \quad 2k_0 = 3 + 1 + \frac{1}{\kappa}.$$

Time-dependent control norms

We introduce the following control norms:

$$A := \|\nabla r - N\|_{L^\infty(\Omega_t)} + \|v\|_{\dot{C}^{\frac{1}{2}}(\Omega_t)},$$

$$B := A + \|\nabla r\|_{\tilde{C}^{\frac{1}{2}}(\Omega_t)} + \|\nabla v\|_{L^\infty(\Omega_t)},$$

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A with $\mathcal{H}^{2k_0}$, $A \lesssim \|(r,v)\|_{\mathbf{H}^{2k}}$, $2k > 2k_0 = 3 + 1 + \frac{1}{\kappa}$.

B with $\mathcal{H}^{2k_0+1}$, $B \lesssim \|(r,v)\|_{\mathbf{H}^{2k}}$, $2k > 2k_0 + 1 = 3 + 2 + \frac{1}{\kappa}$.

Theorem: continuation criterion

For each integer $k \geq 0$ there exists an energy functional $E^{2k} = E^{2k}(r, v)$ with the following properties:

- Coercivity: as long as A remains bounded, we have

$$E^{2k}(r, v) \approx \|(r, v)\|^2_{\mathcal{H}^{2k}}.$$

- Energy estimates hold for solutions, i.e.,

$$\frac{d}{dt} E^{2k}(r, v) \lesssim_A B \|(r, v)\|^2_{\mathcal{H}^{2k}}.$$

Therefore,

$$\|(r, v)\|^2_{\mathcal{H}^{2k}} \lesssim e^{\int_0^t C(A) B(\tau)\, d\tau} \|(\mathring{r}, \mathring{v})\|^2_{\mathcal{H}^{2k}}.$$

In particular, the $\mathbf{H}^{2k}$ solutions can be continued as long as A remains bounded and $B \in L^1_t(\Omega)$.

Previous results: liquid

- A priori estimates: Oliynyk ('17) and Ginsberg ('19).
- Local existence and uniqueness: Oliynyk ('19) and Miao, Shahshahani, and Wu ('20): "hard phase".
- First-order symmetric hyperbolic approach for local existence and uniqueness: Trakhinin ('09).

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Recently, good progress in the case of a liquid.

Previous results: gas under symmetry

- Tolman, Oppenheimer, and Volkoff, TOV eqs ('30s): modeling of a star; spherically symmetric static solutions to the Einstein-Euler system for a fluid body in vacuum; vanishing of the pressure as the correct physical condition on the boundary (gas and liquid).
- Spherically symmetric static solutions of Einstein-Euler: Lindblom ('88), Rendall and Schmidt ('91), Makino ('98).
- Spherical and axisymmetry: Makino ('16, '17): spherical symmetric solutions of Einstein-Euler with a vacuum boundary near equilibrium (TOV). Makino ('18): axisymmetry and slowly rotating (related: Heilig ('95) and Makino ('19)). Hadžić and Lin ('20): "turning point principle" for relativistic stars.

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Much can be said under symmetry assumptions (including considering coupling to Einstein's equations).

Previous results: gas, no physical vacuum

- Existence and uniqueness for a gas not satisfying the physical boundary condition: Rendall ('92), Makino and Ukai ('95), LeFloch and Ukai ('09), Brauer and Karp ('14). (Unnatural regularity near the boundary; no acceleration on the boundary).
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A few results have been obtained if one violates the physical vacuum condition.

Previous results: gas, physical vacuum

- A priori estimates: Jang, LeFloch, and Masmoudi ('16), and Hadžić, Shkoller, and Speck ('19). (Results all in Lagrangian coordinates; we work entirely in Eulerian coordinates.)
- Local existence and uniqueness in $1+1$ dimensions: Oliynyk ('12) (Geometry plays no role, main difficulties are absent).

The linerized equation and D_t

The linearized system for (s, w) $(r \rightsquigarrow s, v \rightsquigarrow w)$:

$$\begin{aligned} D_t s + \frac{1}{\kappa} G^{ij} \partial_i r w_j + r G^{ij} \partial_i w_j + r v^i \partial_i s &= error, \\ D_t w_i + \partial_i s &= error, \end{aligned} \tag{3}$$

admits the energy $\|(s, w)\|_{\mathcal{H}}^2 := \int_{\Omega_t} r^{\frac{1-\kappa}{\kappa}} (s^2 + r G^{ij} w_j w_i) \, dx.$

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Not the case if differentiating w.r.t. D_t because $D_t r \sim r \partial(r, v)$.

The good linearized variables

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$$(w_N)_i := D_t^N v_i,$$
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(s_N, w_N) are good approximations for solutions of the linearized equation.

$$\text{Control of the energy: } E^{2k}(r,v) := \sum_{N=0}^{k} \|(s_{2N}, w_{2N})\|_{\mathcal{H}}.$$

Full control of (r,v): use the equations to get $D_t^{2N} r \sim r^N \partial^{2N} r$, $D_t^{2N} v \sim r^N \partial^{2N} v$. (Recall the $\mathcal{H}^{2k}$ norm).

Second-order transition operators

To establish $D_t^{2N}(r,v) \sim r^N \partial^{2N}(r,v)$ we use elliptic estimates for:

$$L_1 s := G^{ij}\left(r\partial_i\partial_j + \frac{1}{\kappa}\partial_i r \partial_j s\right),$$
$$(L_2 w)_i := G^{ml}\left(\partial_i(r\partial_m w_l) + \frac{1}{\kappa}\partial_m r \partial_i w_l\right),$$

for which $\quad s_{2N} = L_1 s_{2N-2} + F_{2N}, \quad w_{2N} = (L_2 w_{2N-2})_i + (G_{2N})_i.$

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This gives **coercivity of the energy:** $E^{2k}(r,v) \gtrsim \|(r,v)\|^2_{\mathcal{H}^{2k}}$.

Energy estimates

To establish

$$\frac{d}{dt}E^{2k}(r,v) \lesssim_A B\|(r,v)\|^2_{\mathcal{H}^{2k}}$$

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We use that the (s_{2N}, w_{2N}) satisfy the linearized equation with good perturbative terms. Good means that the pertrubative terms can be interpolated with a single factor of B to be controlled by

$$B\|(r,v)\|^2_{\mathcal{H}^{2k}}.$$

Construction of solutions

We construct solutions with a time discretization given by the following steps:

- Regularization.
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To control the iteration, we need to translate our energy estimates to purely spatial estimates at a fixed time. We do so by reinterpreting the operators D_t^N as differential operators at fixed time obtained by reiterating the equations.

Uniqueness

Consider two solutions (r_1, v_1) and (r_2, v_2) defined in Ω_1 and Ω_2. Put $\Omega_t := \Omega_{t,1} \cap \Omega_{t,2}$, $\Gamma_t := \partial \Omega_t$. (Eulerian coordinates.)

Consider the following distance functional (inspired by the linearized eqs.):

$$\mathcal{D}_{\mathcal{H}}((r_1, v_1), (r_2, v_2)) := \int_{\Omega_t} (r_1 + r_2)^{\frac{1-\kappa}{\kappa}} ((r_1 - r_2)^2 + (r_1 + r_2)|v_1 - v_2|^2)\, dx.$$

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- $\int_{\Gamma_t} |r_1 + r_2|^{\frac{1}{\kappa}+2} d\sigma \lesssim \mathcal{D}_{\mathcal{H}}((r_1, v_1), (r_2, v_2))$.

Some questions

Some natural follow-up questions are the following:

- Inclusion of entropy.
- Expanding relativistic gas (Jang, Hadžić, Rickard,...).
- Non-relativistic limit.
- Other models (MHD, gravity, viscosity,...).

— Thank you for your attention —