
TRANSVERSE LINEAR INSTABILITY OF LINE PERIODIC TRAVELING WAVES FOR WATER WAVE MODELS

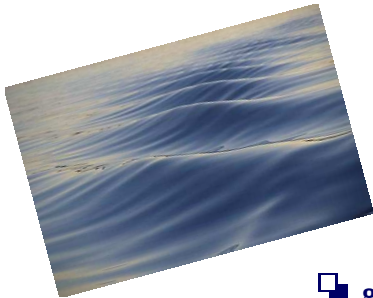
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MSRI PROGRAM MATHEMATICAL PROBLEMS IN FLUID DYNAMICS

PERIODIC WATER WAVES

WATER-WAVE PROBLEM



■ gravity-capillary waves

- *three-dimensional inviscid fluid layer*
- *constant density*
- *gravity and surface tension*
- *irrotational flow*

EULER EQUATIONS

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \quad \text{for} \quad 0 < y < 1 + \eta$$

$$\phi_y = 0 \quad \text{on} \quad y = 0$$

$$\phi_y = \eta_t + \eta_x + \eta_x \phi_x + \eta_z \phi_z \quad \text{on} \quad y = 1 + \eta$$

$$\phi_t + \phi_x + \frac{1}{2} (\phi_x^2 + \phi_y^2 + \phi_z^2) + \alpha \eta - \beta \mathcal{K} = 0 \quad \text{on} \quad y = 1 + \eta$$

- velocity potential ϕ ; free surface $h + \eta$
- mean curvature $\mathcal{K} = \left[\frac{\eta_x}{\sqrt{1+\eta_x^2+\eta_z^2}} \right]_x + \left[\frac{\eta_z}{\sqrt{1+\eta_x^2+\eta_z^2}} \right]_z$
- parameters ρ, g, σ, h

$$\alpha = \frac{gh}{c^2}, \quad \beta = \frac{\sigma}{\rho hc^2}$$

EULER EQUATIONS

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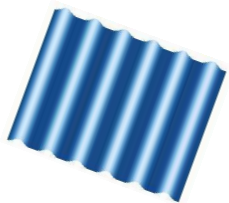
■ very rich dynamics

- symmetries, Hamiltonian structures
- many particular solutions

■ difficulties

- variable domain (free surface)
- nonlinear boundary conditions

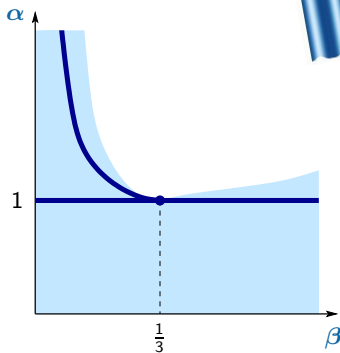
FOCUS ON ...



- TRAVELING PERIODIC 2D WAVES
- TRANSVERSE LINEAR INSTABILITY
- ANALYTICAL RESULTS
- EULER EQUATIONS / LONG-WAVE MODELS

2D PERIODIC TRAVELING WAVES

EXISTENCE RESULTS

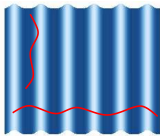


[Nekrasov, Levi-Civita, Struik, Lavrentiev, Friedrichs & Hyers, ...
Amick, Kirchgässner, Iooss, Buffoni, Groves, Toland, Lombardi, Sun,
...]

TRANSVERSE LINEAR (IN)STABILITY

3D PERTURBATIONS

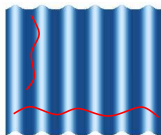
■ perturbations depend on both horizontal coordinates



- **transverse direction z :** *bounded (periodic)*
- **longitudinal direction x :**

3D PERTURBATIONS

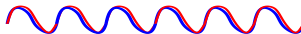
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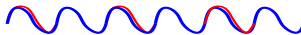
■ **transverse direction z :** *bounded (periodic)*

■ **longitudinal direction x :**

co-periodic perturbations



subharmonic perturbations

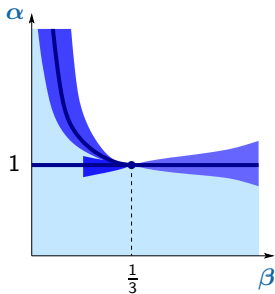


localized/bounded perturbations



PREDICTIONS

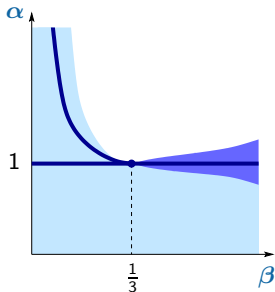
- ■ **long-wave models:** *transverse linear instability*
perturbations: co-periodic in x , periodic in z



PREDICTIONS

■ long-wave models: *transverse linear instability*

perturbations: co-periodic in x , periodic in z



■ large surface tension:

Kadomtsev-Petviashvili I

$$u_{xt} = (u_{xx} + u + u^2)_{xx} - u_{yy}$$

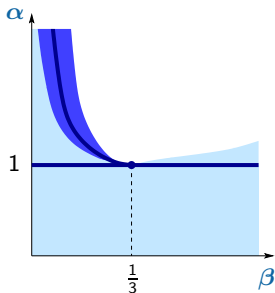
[Johnson & Zumbrun (2010), H. (2011), Hakkaev, Stanislavova & Stefanov (2012), Rousset & Tzvetkov (2012), ...]

Euler equations [H. (2015)]

PREDICTIONS

■ long-wave models: *transverse linear instability*

perturbations: co-periodic in x , periodic in z



■ weak surface tension:

Davey-Stewartson (E-E, focusing)

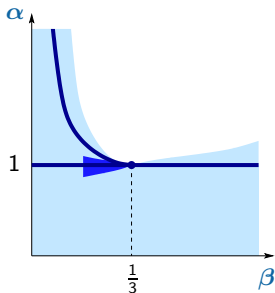
$$\begin{aligned}i\mathbf{A}_t + \mathbf{A}_{xx} + \mathbf{A}_{yy} + (\gamma_1|\mathbf{A}|^2 + \gamma_2\phi_x)\mathbf{A} &= 0 \\ \gamma_3\phi_{xx} + \phi_{yy} - \gamma_3(|\mathbf{A}|^2)_x &= 0\end{aligned}$$

[Godey (2016)]

PREDICTIONS

■ long-wave models: *transverse linear instability*

perturbations: co-periodic in x , periodic in z



■ critical surface tension:

5th order KP equation

$$\mathbf{u}_{xt} = (\mathbf{u}_{xxxx} + \mathbf{u}_{xx} + \mathbf{u}^2)_{xx} + \mathbf{u}_{yy}$$

[H. & Wahlén (2017)]

A COMMON APPROACH

[M. Groves, H., & S.-M. Sun (2001),, Godey (2016)]

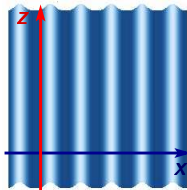
TRANSVERSE INSTABILITY PROBLEM

■ Transverse spatial dynamics

$$U_z = DU_t + F(U)$$

■ $U(x, z, t)$, D linear operator, F nonlinear map

■ a periodic wave $U_*(x)$ is an equilibrium: $F(U_*) = 0$



TRANSVERSE LINEAR INSTABILITY

▣ Transverse spatial dynamics

$$U_z = DU_t + F(U)$$

$U_*(x)$ is **transversely linearly unstable** if the linearized system

$$U_z = DU_t + \mathcal{L}U, \quad \mathcal{L} = F'(U_*)$$

possesses a solution of the form $U(x, z, t) = e^{\lambda t} V_\lambda(x, z)$

with $\lambda \in \mathbb{C}$, $\text{Re}\lambda > 0$, V_λ **bounded function**.

GENERAL INSTABILITY CRITERION

HYPOTHESES

- 1 the system $U_z = DU_t + F(U)$ is **reversible**;
 - 2 the linear operator $\mathcal{L} = F'(U_*)$ possesses a **pair of simple purely imaginary eigenvalues $\pm i\kappa_*$** ;
 - 3 the operators \mathbf{D} and \mathcal{L} are closed in \mathcal{X} with $D(\mathcal{L}) \subset D(\mathbf{D})$;
-

MAIN RESULT

THEOREM

- ① For any $\lambda \in \mathbb{R}$ sufficiently small, the linearized system

$$U_z = DU_t + \mathcal{L}U$$

possesses a solution of the form $U(\cdot, z, t) = e^{\lambda t} V_\lambda(\cdot, z)$

with $V_\lambda(\cdot, z) \in D(\mathcal{L})$ a periodic function in z .

- ② U_* is transversely linearly unstable.

[Godey (2016)]

PROOF OF 1.

□ look for **solutions periodic in z** of

$$U_z = (\lambda D + \mathcal{L})U$$

□ the linear operator $\lambda D + \mathcal{L}$ possesses two complex conjugated purely imaginary eigenvalues

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$$U_z = (\lambda D + \mathcal{L})U$$

□ the linear operator $\lambda D + \mathcal{L}$ possesses two complex conjugated purely imaginary eigenvalues

- \mathcal{L} possesses two simple eigenvalues $\pm i\kappa_*$;
- $\lambda D + \mathcal{L}$ is a small relatively bounded perturbation of \mathcal{L} , for small and real λ ;
- the spectrum of $\lambda D + \mathcal{L}$ is symmetric with respect to the imaginary axis (due to reversibility) and to the real axis (since λ is real).



EULER EQUATIONS

[H., T. Truong & E. Wahlén (2021)]

EULER EQUATIONS

- Hamiltonian formulation of the 3D problem:

$$U_z = DU_t + F(U)$$

- $U = (\eta, \omega, \phi, \xi)$ and $Y = y/(1 + \eta)$
- boundary conditions

$$\phi_y = b(U)_t + g(U) \quad \text{on } y = 0, 1$$

- the system is reversible

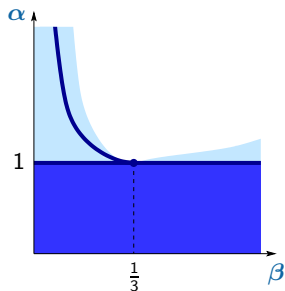
[Groves, H., & Sun (2001), Groves, Sun, & Wahlén (2016)]

[Groves & Mielke (2001)]

2D PERIODIC WATER WAVES

- A one-parameter family of symmetric periodic waves

$\beta > 0, \alpha \in (0, 1)$ linearly transversely unstable



$$\eta(x) = \eta_\varepsilon(k_\varepsilon x), \quad \phi(x, y) = \phi_\varepsilon(k_\varepsilon x, y)$$

- $\eta_\varepsilon, \phi_\varepsilon$ are 2π -periodic in $X = k_\varepsilon x$
- $k_\varepsilon = k_* + k_2 \varepsilon^2 + \mathcal{O}(\varepsilon^4)$
 $\eta_\varepsilon = \varepsilon \eta_1 + \varepsilon^2 \eta_2 + \mathcal{O}(\varepsilon^3)$
 $\phi_\varepsilon = \varepsilon \phi_1 + \varepsilon^2 \phi_2 + \mathcal{O}(\varepsilon^3)$
- explicit formulas for k_*, η_1, ϕ_1 ,
and then also for $k_2, \eta_2, \phi_2, \dots$

LINEARIZED SYSTEM

- linearized system (after rescaling $X = k_\epsilon X$)

$$U_z = DU_t + DF(u_\epsilon)U$$

- boundary conditions

$$\phi_y = Db(u_\epsilon)U_t + Dg(u_\epsilon)U \quad \text{on } y = 0, 1$$

LINEARIZED SYSTEM

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■ boundary conditions

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■ linear operator $\mathcal{L}_\varepsilon := DF(u_\varepsilon)$

■ boundary conditions

$$\phi_y = Dg(u_\varepsilon)U \quad \text{on } y = 0, 1$$

■ space of 2π -periodic functions

$$\mathcal{X} = H^1(0, 2\pi) \times L^2(0, 2\pi) \times H^1((0, 2\pi) \times (0, 1)) \times L^2((0, 2\pi) \times (0, 1))$$

LINEAR OPERATOR \mathcal{L}_ε

$$\boxed{\mathcal{L}_\varepsilon = \mathcal{L}_0 + \mathcal{L}_\varepsilon^1} \quad \mathcal{L}_0 \begin{pmatrix} \eta \\ \omega \\ \phi \\ \xi \end{pmatrix} = \begin{pmatrix} \frac{\omega}{\beta} \\ -k_\varepsilon^2 \beta \eta_{xx} + \alpha \eta - k_\varepsilon \phi_x|_{y=1} \\ \xi \\ -k_\varepsilon^2 \phi_{xx} - \phi_{yy} \end{pmatrix}, \quad \mathcal{L}_\varepsilon^1 \begin{pmatrix} \eta \\ \omega \\ \phi \\ \xi \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \\ G_1 \\ G_2 \end{pmatrix}$$

$$g_1 = \frac{(1 + k_\varepsilon^2 \eta_{\varepsilon x}^2)^{1/2}}{\beta} \left(\omega + \frac{1}{1 + \eta_\varepsilon} \int_0^1 y \phi_{\varepsilon y} \xi \, dy \right) - \frac{\omega}{\beta}$$

$$g_2 = \int_0^1 \left\{ k_\varepsilon^2 \phi_{\varepsilon x} \phi_x - \frac{\phi_{\varepsilon y} \phi_y}{(1 + \eta_\varepsilon)^2} + \frac{\phi_{\varepsilon y}^2 \eta}{(1 + \eta_\varepsilon)^3} - \frac{k_\varepsilon^2 y^2 \eta_{\varepsilon x} \phi_{\varepsilon y} \phi_y}{(1 + \eta_\varepsilon)^2} - \frac{k_\varepsilon^2 y^2 \eta_{\varepsilon x} \phi_{\varepsilon y}^2 \eta_x}{(1 + \eta_\varepsilon)^2} + \frac{k_\varepsilon^2 y^2 \eta_{\varepsilon x}^2 \phi_{\varepsilon y}^2 \eta}{(1 + \eta_\varepsilon)^3} \right. \\ \left. + \left[k_\varepsilon^2 y \phi_{\varepsilon y} \phi_x + k_\varepsilon^2 y \phi_{\varepsilon x} \phi_y - \frac{2k_\varepsilon^2 y^2 \eta_{\varepsilon x} \phi_{\varepsilon y} \phi_y}{1 + \eta_\varepsilon} - \frac{k_\varepsilon^2 y^2 \phi_{\varepsilon y}^2 \eta_x}{1 + \eta_\varepsilon} + \frac{k_\varepsilon^2 y^2 \eta_{\varepsilon x} \phi_{\varepsilon y}^2 \eta}{(1 + \eta_\varepsilon)^2} \right]_x \right\} dy \\ + k_\varepsilon^2 \beta \eta_{xx} - k_\varepsilon^2 \beta \left[\frac{\eta_x}{(1 + k_\varepsilon^2 \eta_{\varepsilon x}^2)^{3/2}} \right]_x$$

$$G_1 = -\frac{\eta_\varepsilon \xi}{1 + \eta_\varepsilon} + \frac{(1 + k_\varepsilon^2 \eta_{\varepsilon x}^2)^{1/2}}{\beta(1 + \eta_\varepsilon)} \left(\omega + \frac{1}{1 + \eta_\varepsilon} \int_0^1 y \phi_{\varepsilon y} \xi \, dy \right) y \phi_{\varepsilon y}$$

$$G_2 = \left[\frac{\eta_\varepsilon \phi}{(1 + \eta_\varepsilon)} + \frac{\phi_\varepsilon \eta}{(1 + \eta_\varepsilon)^2} \right]_{yy} - k_\varepsilon^2 [\eta_\varepsilon \phi_x + \phi_{\varepsilon x} \eta - y \phi_{\varepsilon y} \eta_x - y \eta_{\varepsilon x} \phi_y]_x \\ + k_\varepsilon^2 \left[y \eta_{\varepsilon x} \phi_x + y \phi_{\varepsilon x} \eta_x + \frac{y^2 \eta_{\varepsilon x}^2 \phi_{\varepsilon y} \eta}{(1 + \eta_\varepsilon)^2} - \frac{y^2 \eta_{\varepsilon x}^2 \phi_y}{1 + \eta_\varepsilon} - \frac{2y^2 \eta_{\varepsilon x} \phi_{\varepsilon y} y \eta_x}{1 + \eta_\varepsilon} \right]_y$$

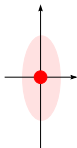
CHECK HYPOTHESES

- **Main step:** *show that \mathcal{L}_ε possesses a pair of simple eigenvalues $\pm i\ell_\varepsilon$, for ε sufficiently small.*
 - *operator with compact resolvent \longrightarrow pure point spectrum*

CHECK HYPOTHESES

■ **Main step:** show that \mathcal{L}_ε possesses a pair of simple eigenvalues $\pm i\ell_\varepsilon$, for ε sufficiently small.

- operator with compact resolvent \longrightarrow pure point spectrum
- **spectrum of \mathcal{L}_0 :** only one purely imaginary eigenvalue

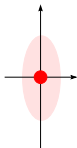


- eigenvalue 0 with geometric multiplicity 3 and algebraic multiplicity 6

CHECK HYPOTHESES

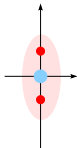
■ **Main step:** show that \mathcal{L}_ε possesses a pair of simple eigenvalues $\pm i\ell_\varepsilon$, for ε sufficiently small.

- operator with compact resolvent \longrightarrow pure point spectrum
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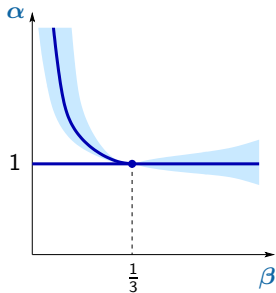
- use a **spectral decomposition** and prove that \mathcal{L}_ε has the eigenvalues:



- eigenvalue 0 with geometric multiplicity 2 and algebraic multiplicity 4
- a pair of simple eigenvalues $\pm i\ell_\varepsilon$

SOME RELATED PROBLEMS

SOLITARY WAVES



■ strong surface tension

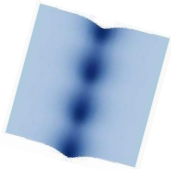
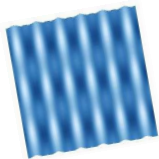
- linear transverse instability
[Groves, H. & Sun (2001), Pego & Sun (2004)]
- nonlinear transverse instability
[Rousset & Tzvetkov (2011)]

■ weak surface tension

- linear transverse instability
[Groves, Sun, & Wahlén (2016)]

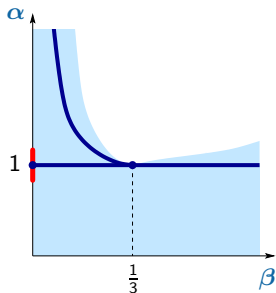
DIMENSION-BREAKING BIFURCATIONS

- existence of **transversely periodically modulated waves**



- **periodic waves:** large surface tension
[H. (2015)]
- **solitary waves:** large and weak surface tension
[Groves, H., & Sun (2001), Groves, Sun, & Wahlén (2016)]

GRAVITY WAVES



- model equation:

Kadomtsev-Petviashvili II equation

$$u_{xt} = (u_{xx} + u + u^2)_{xx} + u_{yy}$$

- prediction: **transverse stability of periodic/solitary waves**

[H., Li & Pelinovsky (2017)]

[Mizumachi & Tzvetkov (2012),

Mizumachi (2015)]



