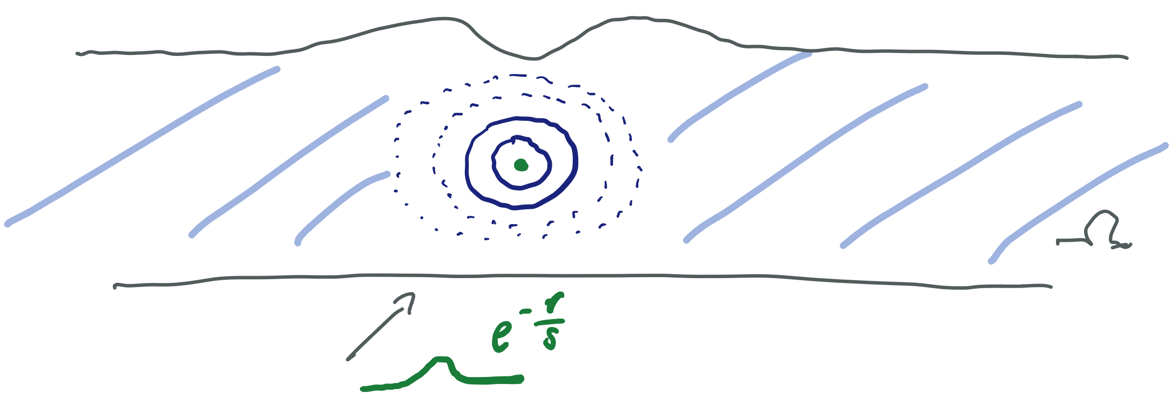
Smooth stationary water waves with exponentially decaying and localised vorticity

joint w. S. Walsh and C. Zeng (JEMS)



Two ingredients:

$\Delta\Psi = \gamma(\Psi)$ Euler

$\Psi = 0$ at $\partial\Omega$

$\frac{1}{2}|\nabla\Psi|^2 + \underline{g}y + \underline{\alpha}^2\kappa = g$

at surface

$\Delta U = \gamma(U)$

$U = U(r) > 0,$

exp. dec. in $\mathbb{R}^2$

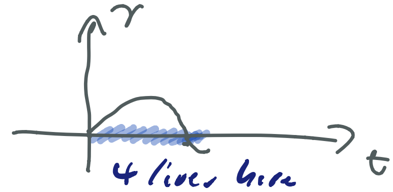
'Ground state'

Our approach: $\Psi = \underbrace{U - \hat{U}_{be}}_{0 \text{ at } \partial\Omega} + \underline{\underline{u}}$ ← the variable

- What is the vorticity γ?

Assumption: $\gamma(t) = t + O(t^2)$, such that $\bar{U}$ exists.

For example: $\gamma(t) = t - ct^2 + \dots$

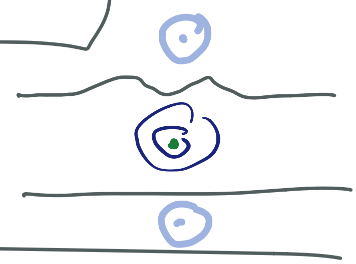


Then ψ exists, explicit leading-order formulas, and:

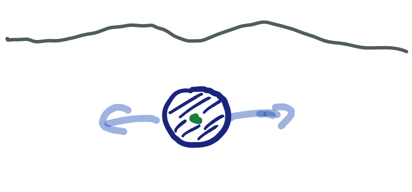
$$|\omega|_\infty \approx \frac{1}{\delta^2}, \quad |\omega|_{L^1} \approx |A\bar{U}|_{L^1}$$

$$\int_\Omega \omega \, dA \approx 0.$$

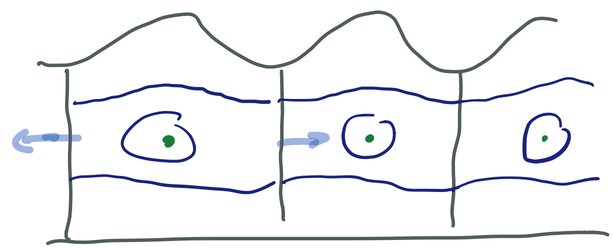
$0 < \delta \ll 1.$



'In between' two different sorts of waves:



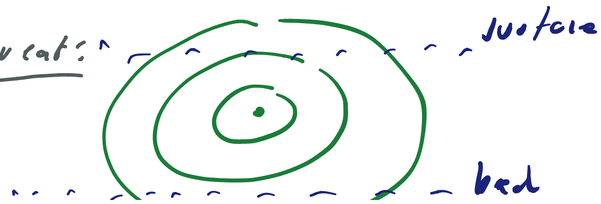
Compactly supported vorticity

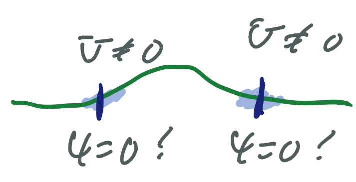


affine vorticity

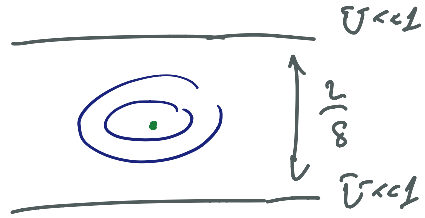
$\gamma(t) = c_1 t + c_2$

Caveat:





Idea: (i) scale $(x,y) \mapsto \frac{(x,y)}{\delta}$

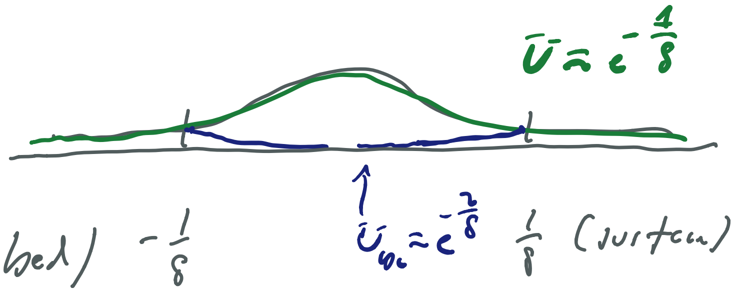


(ii) Use γ as a guide for a boundary correction:

$$\Delta \bar U = \gamma(\bar U) \xrightarrow{\gamma \to \infty} \bar U, \text{ so let}$$

$$\sim \bar U + O(\bar U^2)$$

$$\boxed{\begin{aligned} \Delta \bar U_{bc} &= \bar U_{bc} \\ \bar U_{bc}\big|_{\partial\Omega} &= \bar U \end{aligned}}$$



The operator $1-\Delta$, instead of $\gamma(\cdot)-\Delta$, is also to the linearised problem:

Expand $\Delta \Psi = \gamma(\Psi)$ around $\bar U$ $(\Psi = \bar U - \bar U_{bc} + u)$

$$\Rightarrow \boxed{(-\Delta + \gamma'(\bar U))u + \text{nonlinear}(u) = 0}$$

Need to invert $-\Delta + \gamma'(\bar U)$ on $H_0^1(\Omega)$.

But: $\Delta \bar U = \gamma(\bar U) \xrightarrow{\partial_y} \Delta \bar U_y = \underbrace{\gamma'(\bar U)}_{\approx 1} \bar U_y$

$$\Rightarrow \boxed{\bar U_y \in \ker(\Delta - \gamma'(\bar U)) \text{ on } \mathbb{R}^2}$$

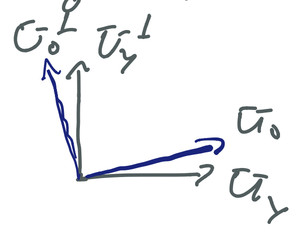
$\Psi \simeq \bar U \Rightarrow$ cannot invert the lin. problem in $\mathbb{R}^2$!

$\bar U_y$ acts as a degenerate direction in the problem.

We write $\bar U_0 = \bar U_y - (\bar U_y)_{bc}$ + extremely small

to get an exact eigenfunction in $H^1(\Omega)$:

$$\boxed{(-\Delta + \gamma'(\bar U))\, \bar U_0 = \lambda \bar U_0}$$



Leads to a diagonalization:

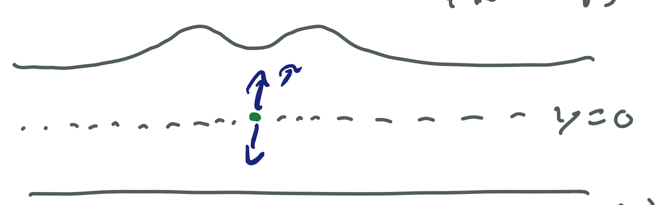
$$\begin{cases} (-\Delta + \gamma'(\bar U))u + \text{nonl.}(u) = 0 \quad \text{in } U_0^\perp \quad ✓ \\ \langle (-\Delta + \gamma'(\bar U))u, U_0 \rangle + \langle \text{nonl.}(u), \bar U_0 \rangle = 0 \\ + \text{ Bernoulli at surface: linearized: } g - \alpha^2 \partial_x^2 \quad ✓ \end{cases}$$

□ boils down to the eigenvalue $\lambda \simeq 0$.

Need $\neq 0$ to solve for u via a contraction/IFT.

$\psi \approx \bar U(\frac{1}{\delta})$

The problem is :



$\psi \approx O(-\frac{1}{\delta})$

All values essentially identical at surface and bed, and of order $\pm e^{-\frac{2}{\delta}}$.

Introduce translation τ:

Linear terms $\approx \operatorname{sgn}(\tau)\, e^{-\frac{(2-|\tau|)}{\delta}}$

$\Longrightarrow$ Diff. eq $\big|_{\tau > \varepsilon} > 0$, Diff. eq $\big|_{\tau < -\varepsilon} < 0$ $\overset{\text{int. med. v.}}{\Longrightarrow}$

$\exists\, \tau_0$: Diff. Eq $= 0$

τ is exceedingly small: $\tau \lessapprox \delta^{-\frac{3}{8}} e^{-\frac{1}{2\delta}}$.

Possible future problems:

- Periodic problem (with surface tension)
- Gravity problem (looks hard)
- Infinite depth? (no chance with $\psi\big|_{\partial\Omega} = 0$)
- Multiple spheres.
