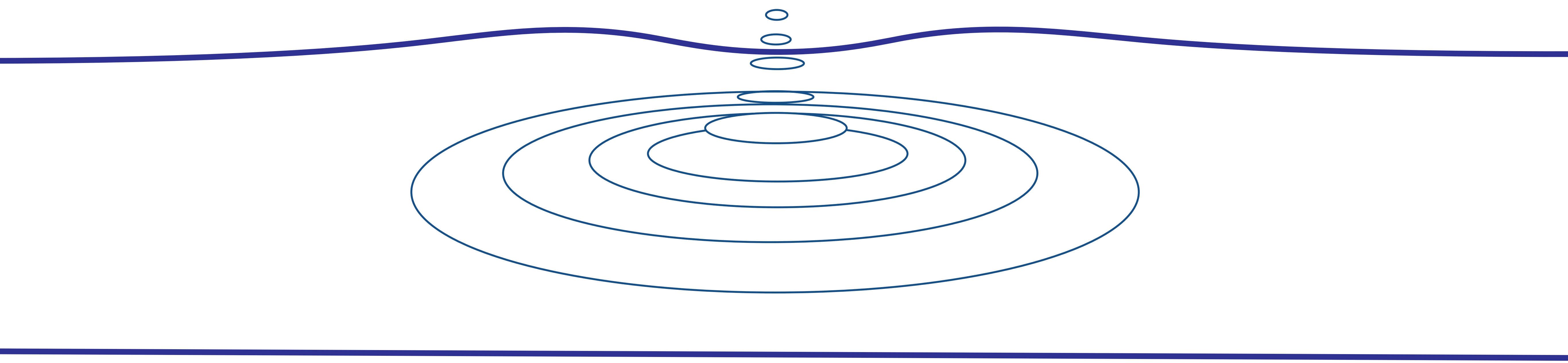
Smooth stationary water waves with exponentially localised vorticity



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Mathematics of Fluid Dynamics program
MSRI, Berkeley, April 2021

Assumptions

A. $\gamma(t) = t + \mathcal{O}(t^2)$ is a C^2-function such that $\Delta U = \gamma(U)$ admits a radial solution in $C^2(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$.

B. $\{\partial_x U, \partial_y U\}$ spans the kernel of $-\Delta + \gamma'(U) \colon H^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$.

Euler

$$\Delta \Psi = \gamma(\Psi)$$
$$\Psi|_{\partial\Omega} = 0$$

+ nonlinear Bernoulli

Ground state

$$\Delta U = \gamma(U)$$
$$U = |_{\partial\mathbb{R}^2} 0$$

+ positivity, decay

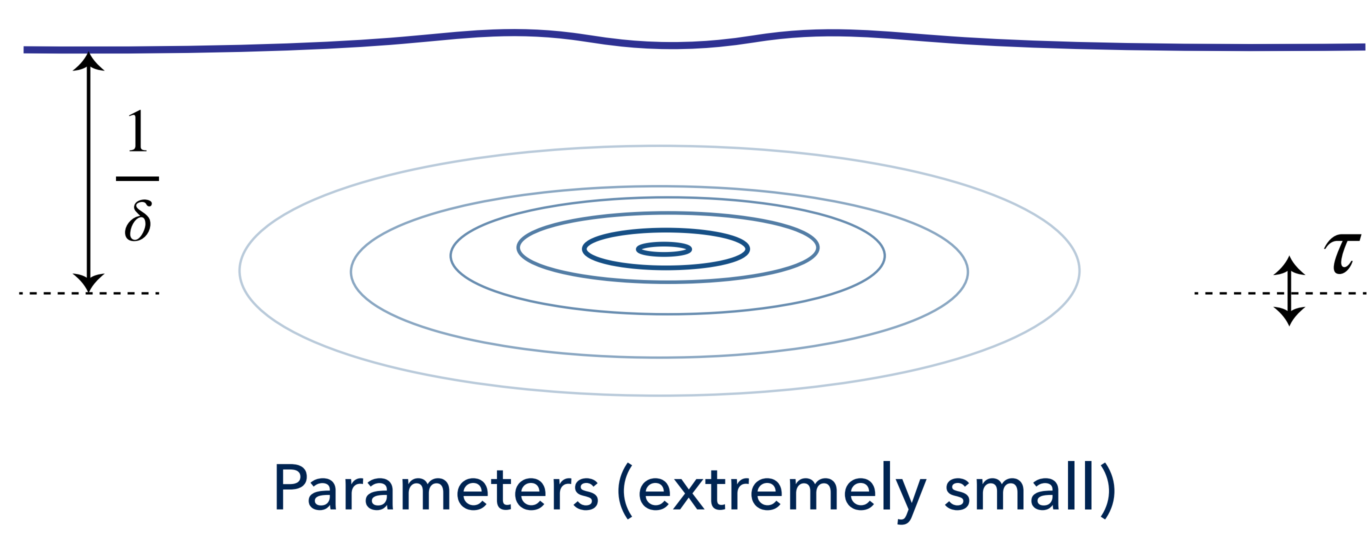
Boundary correction

$$\Delta U_{\text{bc}} = U_{\text{bc}}$$
$$U_{\text{bc}}|_{\partial\Omega} = U$$

acts as a reflection around boundaries

Stream function ansatz

$$\Psi = U - U_{\text{bc}} + u$$



Main result

For small $\delta \ll 1$, there exists $|\tau| \lesssim \delta^{-\frac{3}{8}} e^{-\frac{1}{2\delta}}$, and a solution with the properties that $|\Psi - \Psi_0|_{H^2(\Omega)} \lesssim \delta^{-3} e^{-\frac{2}{\delta}}$, where

$$\Psi_0(x,y) = U\left(\frac{x}{\delta}, \frac{y-\tau}{\delta}\right) - U\left(\frac{x}{\delta}, \frac{2-y-\tau}{\delta}\right) - U\left(\frac{x}{\delta}, \frac{-2-y-\tau}{\delta}\right),$$

and $|\eta - \eta_0|_{H^2(\mathbb{R})} \lesssim \delta^{-\frac{5}{4}} e^{-\frac{3}{\delta}}$, with $\eta_0 = -\frac{4}{\alpha\sqrt{g}\delta^2} e^{-\frac{\sqrt{g}}{\alpha}|\cdot|} * \left(\left(\partial_y U(\frac{\cdot}{\delta}, \frac{1}{\delta})\right)^2\right)$.

The **vorticity is spiked** in the sense that $\omega = \frac{1}{\delta^2}\gamma\left(U(\frac{\cdot}{\delta})\right) + o(e^{-\frac{1}{2\delta}})$, so that

$$|\omega|_{L^\infty(\Omega)} = O\left(\delta^{-2}\right), \qquad |\omega|_{L^1(\Omega)} = |\Delta U|_{L^1(\mathbb{R}^2)} + o(e^{-\frac{1}{2\delta}}).$$

At the same time, $\int_\Omega \omega \, \mathrm{d}x = o(e^{-\frac{1}{2\delta}})$.

Thank you for your attention!