VARIOUS BOUNDARY CONDITIONS FOR THE STOKES OPERATOR IN NON SMOOTH DOMAINS

Sylvie MONNIAUX
Aix-Marseille, FRANCE

MSRI - March 11th, 2021
Euler / Navier-Stokes seminar

Data of the talk

$\Omega \subset \mathbb{R}^3$ bounded Lipschitz domain

$\nu(x)$: outer unit normal at a point x of the boundary $\partial\Omega$

Unknowns

$u : [0,T] \times \Omega \longrightarrow \mathbb{R}^3$ velocity of the fluid

$\pi : [0,T] \times \Omega \longrightarrow \mathbb{R}$ pressure

Initial condition

$u_0 : \Omega \longrightarrow \mathbb{R}^3$ initial velocity with $\operatorname{div} u_0 = 0$

Equations

(NS)
$$\partial_t u - \Delta u + \nabla\left(\pi + \tfrac{1}{2}|u|^2\right) = \left|\begin{array}{l} u \times \operatorname{curl} u \\ -(u\cdot\nabla)u \end{array}\right. \quad \text{in } (0,T)\times\Omega$$
$$\operatorname{div} u = 0$$

$$\boxed{(u\cdot\nabla)u = \tfrac{1}{2}\nabla|u|^2 - u \times \operatorname{curl} u}$$

BOUNDARY CONDITIONS

"no slip bc" (homogeneous Dirichlet bc) $u = 0$ on $(0,T) \times \partial\Omega$

[i.e. $\nu \cdot u = 0$ and $\nu \times u = 0$]

Neumann bc : $(\lambda \nabla u + (\nabla u)^T)\nu - \pi \nu = 0$ on $(0,T) \times \partial\Omega$

$(\lambda \in (-1,1])$ if $\lambda = 0$: $\partial_\nu u - \pi \nu = 0$

if $\lambda = 1$: $\nu \cdot (\nabla u + (\nabla u)^T)\nu = \pi$, $[(\nabla u + (\nabla u)^T)\nu]_{tan} = 0$

"perfectly conducting wall condition" (Hodge bc) $\nu \cdot u = 0$, $\nu \times \operatorname{curl} u = 0$ on $(0,T) \times \partial\Omega$

Remark If $\partial\Omega$ is $C^{1,1}$:

$$[(\nabla u + (\nabla u)^T)\nu]_{tan} = -\nu \times \operatorname{curl} u + 2\mathcal{W} u$$

$\mathcal{W}$: Weingarten map

$\mathcal{W} X = -\nabla_X \nu$

$\mathcal{W} = 0$ on flat portion of $\partial\Omega$

MILD SOLUTIONS - CRITICAL SPACES

Convert (NS) into an integral equation

$$u(t) = e^{-tA} u_0 + \int_0^t e^{-(t-s)A} \, \mathbb{P} \begin{pmatrix} -\nabla \cdot (u(s) \otimes u(s)) \\ \text{or} \\ u(s) \times \operatorname{curl} u(s) \end{pmatrix} ds \qquad 0 < t < T$$

for a suitable operator A (negative generator of a semigroup)
an a suitable projection $\mathbb{P}$ (onto divergence-free vector fields)

Look for solutions $u \in \mathcal{C}([0,T]; X)$

with $X = L^3, H^{1/2}$

Lebesgue Sobolev

METHOD : Picard iterative scheme

Relies on the analyticity of the semigroup $(e^{-tA})_{t\geq 0}$

and the description of fractional powers $(A^{\alpha})_{0<\alpha<1}$,

as well as the properties of the projection $\mathbb{P}$

The problem becomes : Find a fixed point of

$$u = a + B(u,u)$$

where a comes from the initial cond. and B is bilinear.

Dirichlet boundary conditions

$\Omega \subset \mathbb{R}^3$ bounded Lipschitz domain

L^2 theory

$$L^2(\Omega,\mathbb{R}^3) \xrightarrow{\mathbb{P}} \mathcal{H} = \{u \in L^2(\Omega,\mathbb{R}^3) : \operatorname{div} u = 0 \text{ in } \Omega \,, \ \nu . u = 0 \text{ on } \partial\Omega\}$$

$$D(A) = \{u \in H^1_0(\Omega,\mathbb{R}^3) \cap \mathcal{H} \,;\ \exists \pi \in L^2(\Omega) : -\Delta u + \nabla\pi \in \mathcal{H}\}$$

- $(e^{-tA})_{t\geq 0}$ is an analytic sg on $\mathcal{H}$
- $D(A^{1/4}) \subset L^3(\Omega,\mathbb{R}^3)$
- $D(A^{3/4}) \subset W^{1,3}(\Omega,\mathbb{R}^3)$

$$\begin{tikzcd}
H^1_0 \arrow[dd, bend right, "-\Delta_D"'] & & H^1_0 \cap \mathcal{H} = V \arrow[ll, "J_0"'] \arrow[dd, bend left, "A"] \\
L^2 \arrow[rr, shift right, "\mathbb{P}"'] & & \mathcal{H} \arrow[ll, shift right, "J"'] \\
H^{-1} \arrow[rr, "\mathbb{P}_1 = J_0'"'] & & V'
\end{tikzcd}$$

<u>Recall</u>: u mild solution of (NS) if $u = a + B(u,u)$

where $a(t) = e^{-tA} u_0$

$$B(u,v)(t) = \int_0^t e^{-(t-s)A} \mathbb{P}\left[\nabla . (u(s) \otimes v(s))\right] ds$$

<u>THEOREM</u> [M. Mitrea, S. Monniaux JFA '08]

- <u>global existence</u> in $\mathcal{C}([0,\infty); D(A^{1/4}))$ for initial condition <u>u_0</u> $\in D(A^{1/4})$ <u>small enough</u>
- <u>local existence</u> for each <u>$u_0 \in D(A^{1/4})$</u> : $\exists\, T > 0$ s.t. there exists a solution in $\mathcal{C}([0,T); D(A^{1/4}))$
- <u>uniqueness</u> : for each $u_0 \in D(A^{1/4})$, there is at most one solution in <u>$\mathcal{C}([0,T), D(A^{1/4}))$</u>

<u>Remark</u> Uniqueness uses maximal regularity :

$$R : f \longmapsto \left[Rf : t \longmapsto \int_0^t e^{-(t-s)A} f(s)\, ds\right]$$

$$\left(\frac{d}{dt}\right)^{\alpha} A^{1-\alpha} R \in \mathcal{L}(L^p_t(L^q_x))$$

$$0 \le \alpha \le 1$$

$$1 < p,q < \infty$$

L^p theory

(1) $\mathbb{P} : L^2(\Omega, \mathbb{R}^3) \longrightarrow \mathcal{H}_2 = \{u \in L^2(\Omega, \mathbb{R}^3) : \operatorname{div} u = 0 \text{ in } \Omega,\ \nu \cdot u = 0 \text{ on } \partial\Omega\}$

extends to a bdd operator from $L^p(\Omega, \mathbb{R}^3)$ to $\mathcal{H}_p$

for $\underbrace{\frac{3+\varepsilon}{2+\varepsilon}} < p < \underbrace{3+\varepsilon}$ [Fabes, Mendez, Mitrea JFA '98]

indep. from dimension

conjecture of M. Taylor (2000)

(2) $(e^{-tA})_{t\geq 0}$ extends to an analytic semigroup on $\mathcal{H}_p$

for $\left|\frac{1}{2} - \frac{1}{p}\right| < \frac{1}{6} + \varepsilon$

$\frac{1}{2d}$ d: dimension

[Z. Shen ARMA '12]

(3) $D(A_p^\alpha) = D((-\Delta_p^D)^\alpha) \cap \mathcal{H}_p$ if $|\alpha| < \frac{1}{2} - \left(\frac{1}{3} + 2\varepsilon\right)^{-1} \left|\frac{1}{2} - \frac{1}{p}\right|$

$\frac{1}{d}$

[Kunstmann, Weis JEE '17]

NEUMANN BOUNDARY CONDITIONS

(L^2 theory only)

Meaning of the boundary condition :

$$\partial_\nu^\lambda (u, \pi) := \left(\lambda \nabla u + (\nabla u)^T\right)\nu - \pi \nu = 0$$

If $v \in L^2(\Omega, \mathbb{R}^3)$ and $\operatorname{div} v \in L^2(\Omega)$, then $\nu . v \in H^{-1/2}(\partial\Omega)$:

$\varphi \in H^{1/2}(\partial\Omega)$: $H^1(\Omega)$ - extension Φ

$${}_{H^{-1/2}}\langle \nu . v, \varphi \rangle_{H^{1/2}} = \int_\Omega v . \nabla \phi + \int_\Omega \operatorname{div} v . \phi$$

(indep. of the choice of the extension)

In our case (vector-valued) : if $\operatorname{div} u = 0$

$$-\Delta u + \nabla \pi = -\operatorname{Div}\left(\lambda \nabla u + (\nabla u)^T - \pi \, Id\right)$$

$$L^2(\Omega,\mathbb{R}^3) = \overset{\mathbb{Q}\text{ projection}}{\mathcal{K}_2} \overset{\perp}{\oplus} \nabla H^1_0(\Omega) \qquad \left(= \mathcal{H} \overset{\perp}{\oplus} \nabla H^1(\Omega)\right)$$

Functional setting

$$\mathcal{K}_2 = \{u \in L^2(\Omega,\mathbb{R}^3) : \operatorname{div} u = 0 \text{ in } \Omega\}$$

Thm [Fabes, Mendez, Mitrea JFA '98] $\mathbb{Q} : L^2(\Omega,\mathbb{R}^3) \longrightarrow \mathcal{K}_2$ extends to a bounded operator from $L^p(\Omega,\mathbb{R}^3)$ to $\mathcal{K}_p$ for every $\frac{3+\varepsilon}{2+\varepsilon} < p < 3+\varepsilon$.

Neumann-Stokes operator B_2

$$D(B_2) = \{u \in H^1(\Omega,\mathbb{R}^3) \cap \mathcal{K}_2 ;\ \exists \pi \in L^2(\Omega) : -\Delta u + \nabla\pi \in \mathcal{K}_2 \text{ and } \partial_\nu^\lambda(u,\pi) = 0 \text{ on } \partial\Omega\}$$

$$B_2 u = -\Delta u + \nabla \pi$$

As in the case of Dirichlet boundary conditions:

- $(e^{-tB_2})_{t\geq 0}$ is an analytic semigroup in $\mathcal{K}_2$
- $D(B_2^{s/2}) = \mathcal{K}_2 \cap \mathcal{H}^s(\Omega, \mathbb{R}^3)$ for all $0 \leq s < 3/2$
- $D(B_2^{3/4}) \subset W^{1,3}(\Omega, \mathbb{R}^3)$

THEOREM [M. Mitrea, S. Monniaux, M. Wright JMS '11]

- global existence for initial value u_0 small enough in $D(B_2^{1/4})$
- local existence in $\mathcal{C}([0,T), D(B_2^{1/4}))$: for each $u_0 \in D(B_2^{1/4})$, there exists $T>0$ and u mild solution
- uniqueness there exists at most one solution in $\mathcal{C}([0,T), D(B_2^{1/4}))$

L^p theory ?

HODGE BOUNDARY CONDITIONS

Differential forms

exterior algebra $\quad \Lambda = \overset{\mathbb{R}}{\overset{\wr}{\Lambda^0}} \oplus \overset{\mathbb{R}^3}{\overset{\wr}{\Lambda^1}} \oplus \overset{\mathbb{R}^3}{\overset{\wr}{\Lambda^2}} \oplus \overset{\mathbb{R}}{\overset{\wr}{\Lambda^3}}$

exterior product : $\wedge$

interior product : $\lrcorner$

inner product : $\langle \cdot , \cdot \rangle : \Lambda^\ell \times \Lambda^\ell \longrightarrow \mathbb{R}$

$a \in \Lambda^1, \; u \in \Lambda^\ell, \; v \in \Lambda^{\ell+1}$:

$$\langle a \wedge u, v \rangle = \langle u, a \lrcorner v \rangle$$

$a \in \mathbb{R}^3$ vector

φ scalar, 0-form : $\quad a \wedge \varphi = a \quad , \quad a \lrcorner \varphi = 0$

3-form : $\quad a \wedge \varphi = 0 \quad , \quad a \lrcorner \varphi = \varphi a$

u vector, 1-form : $\quad a \wedge u = a \times u \quad , \quad a \lrcorner u = a \cdot u$

2-form : $\quad a \wedge u = a \cdot u \quad , \quad a \lrcorner u = - a \times u$

exterior derivative : $\quad d := \nabla \wedge = \sum_{j=1}^{3} \partial_j e_j \wedge$

d acts on $L^2(\Omega,\Lambda)$ as follows : (unbounded operator)

$$d: \; L^2(\Omega,\Lambda^0) \xrightarrow{\nabla} L^2(\Omega,\Lambda^1) \xrightarrow{\mathrm{curl}} L^2(\Omega,\Lambda^2) \xrightarrow{\mathrm{div}} L^2(\Omega,\Lambda^3)$$

$$\xleftarrow[\substack{\text{with b.c.}\\ \nu\cdot u=0}]{-\mathrm{div}} \qquad \xleftarrow[\substack{\text{with b.c.}\\ \nu\times u=0}]{\mathrm{curl}} \qquad \xleftarrow[\substack{\text{with b.c.}\\ \varphi|_{\partial\Omega}=0}]{-\nabla} \quad : d^*$$

adjoint of d

Hodge-Dirac operator : $D = d + d^*$ with domain

$$D(d)\cap D(d^*) = \{u\in L^2(\Omega,\Lambda) : du \in L^2(\Omega,\Lambda) \;\&\; \underbrace{d^*u}_{\text{b.c.}} \in L^2(\Omega,\Lambda)\}$$

$(d^2 = d\circ d = 0)$

Hodge-Laplacian : $D^2 = (d+d^*)^2 = dd^* + d^*d = -\Delta_H$

on 0-forms : Neumann Laplacian – on 3-forms : Dirichlet-Laplacian

on 1-forms : $D^2 = \boxed{\mathrm{curl}\,\mathrm{curl} - \nabla\,\mathrm{div} = -\Delta}$ – on 2-forms : $D^2 = \mathrm{curl}\,\mathrm{curl} - \nabla\,\mathrm{div} = -\Delta$

with b.c. $\nu\cdot u = 0$, $\nu\times \mathrm{curl}\,u = 0$ $\partial\Omega$ — with b.c. $\nu\times u = 0$, $\mathrm{div}\,u = 0$ $\partial\Omega$

Remark The space $\mathcal{H}_2 = \{u \in L^2(\Omega, \mathbb{R}^3) : \operatorname{div} u = 0 \text{ in } \Omega \ \& \ \nu \cdot u \text{ on } \partial\Omega\}$ corresponds to $N(d^*)|_{\Lambda^1}$: nullspace of d^* restricted to 1-forms

Hodge decomposition

$$\begin{tikzcd} L^2(\Omega,\Lambda) \arrow[r, phantom, "="] \arrow[r, bend right, "\mathbb{Q}"'] \arrow[rr, bend left, "\mathbb{P}"] & R(d) \arrow[r, phantom, "\overset{\perp}{\oplus}"] & N(d^*) \end{tikzcd}$$

p-version : $\qquad L^p(\Omega, \Lambda) = R^p(d) \oplus N^p(d^*)$

OK for $p_H < p < p^H$

($p^H = p_H' > 3$ if Ω is a bounded Lipschitz domain

$p_H = 1$, $p^H = \infty$ if Ω is smooth)

⚠ $D(d) \cap D(d^*) = \{u \in L^2 : du \in L^2, d^*u \in L^2\}$ is <u>not</u> a subspace of $\mathcal{H}^1$

(not even a subspace of $\mathcal{H}^{1/2+\varepsilon}$) but $Tr|_{\partial\Omega} : D(d) \cap D(d^*) \to L^2(\partial\Omega)$ is compact

[C. Denis, AdM '20]

Following results come from joint works with M. Mitrea (2009), S. Hofmann & M. Mitrea (2011), A. McIntosh (2018)

THEOREM

Ω bounded Lipschitz domain in $\mathbb{R}^3$

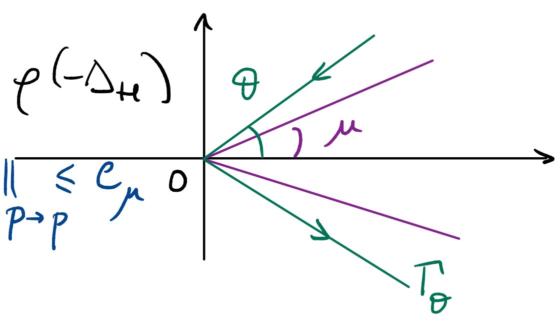
(1) $p_H < p < p^H$:

$-\Delta_H$ is sectorial of angle 0 in $L^p(\Omega, \Lambda)$ holomorphic semigroup

$\forall\, \mu \in (0, \frac{\pi}{2})$, $-\Delta_H$ admits a bounded $S^\circ_{\mu+}$ holomorphic functional calculus in $L^p(\Omega,\Lambda)$

maximal regularity

$$\|\lambda(\lambda I+\Delta_H)^{-1}\|_{p\to p} \le C_\mu$$



$$f(-\Delta_H) = \frac{1}{2\pi i}\int_{\Gamma_\theta} f(z)\,(zI+\Delta_H)^{-1}\,dz$$

(2) The Hodge-Stokes operator $S = d^*d|_{\Lambda^1}$ in $N(d^*)$ is sectorial of angle 0 in $N^p(d^*)|_{\Lambda^1}$ $\forall\ 1 < p < p^H = 3+\varepsilon$

and for $1 < p < p^H$, $p \le q < p^H$ and α s.t. $\frac{1}{p} - \frac{\alpha}{3} = \frac{1}{q}$:

$$\sup_{t \ge 0} \| t^{\alpha/2} e^{-tS} \|_{p \to q} + \sup_{t \ge 0} \| t^{\frac{1+\alpha}{2}} d\, e^{-tS} \|_{p \to q} < \infty$$

(3) global existence of solutions for (NS) if $\|u_0\|_3$ small enough
local existence for each $u_0 \in N^3(d^*)|_{\Lambda^1}$

Remark

$$B(u,v)(t) = \int_0^t e^{-(t-s)S}\, \mathbb{P} \underbrace{\left(u(s) \times \operatorname{curl} u(s)\right)}_{u(s) \lrcorner\, du(s)} ds$$

PERSPECTIVES

• Boussinesq system in $\mathbb{R}^3$

$$\begin{cases} \partial_t u - \Delta u + \nabla \pi + (u.\nabla) u = \theta e_3 \\ \operatorname{div} u = 0 \\ \partial_t \theta - \Delta \theta + (u.\nabla)\theta = 0 \end{cases}$$

[joint work with Lorenzo BRANDOLESE]

uniqueness $(u,\theta) \in \mathcal{C}([0,T], L^3) \times L^2(0,T; L^{3/2})$ via maximal regularity

• MHD with Hodge boundary conditions

$$\begin{cases} \partial_t u - \Delta u + \nabla \pi - u \times \operatorname{curl} u = \operatorname{curl} b \times b, & \operatorname{div} u = 0 \\ \partial_t b - \Delta b = \operatorname{curl}(u \times b), & \operatorname{div} b = 0 \end{cases}$$

$$\left.\begin{array}{ll} \nu \cdot u = 0, & \nu \times \operatorname{curl} u = 0 \\ \nu \times b = 0, & \operatorname{div} b = 0 \end{array}\right\} \text{ on } \partial\Omega$$

(in progress)

THANK YOU FOR YOUR ATTENTION