Non-Conservative $H^{\frac{1}{2}-}$ Weak Solutions of the Incompressible 3D Euler Equations

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MSRI Euler-Navier-Stokes Seminar
Feb 18, 2021

Main Result

We consider the incompressible 3D Euler equations for an unknown velocity $u : \mathbb{R} \times \mathbb{T}^3 \to \mathbb{R}^3$ and pressure $p : \mathbb{R} \times \mathbb{T}^3 \to \mathbb{R}$

$$\begin{aligned} \partial_t u + \operatorname{div}(u \otimes u) + \nabla p &= 0 \\ \operatorname{div} u &= 0 \end{aligned}$$

Theorem

Fix $\beta \in (0, 1/2)$. For any divergence-free, mean-zero vector fields $v_{\text{start}}, v_{\text{end}} \in L^2(\mathbb{T}^3)$, any $T > 0$, and any $\varepsilon > 0$, there exists a weak solution $v \in C([0, T]; H^\beta(\mathbb{T}^3))$ to the 3D Euler equations such that $\|v(0) - v_{\text{start}}\|_{L^2}, \|v(T) - v_{\text{end}}\|_{L^2} < \varepsilon$.

Motivation: K41, intermittency (see Vlad's talk!)
Goal of the talk: (1) - Setup and Main Difficulties, (2) - Key heuristic estimates, (3) - Putting everything together

Setup and Main Difficulties: Inductive Assumptions

- Induction on q: $(u_q, \mathring{R}_q)$ solve the Euler-Reynolds system

$$\begin{aligned} \partial_t u_q + \operatorname{div}(u_q \otimes u_q) + \nabla p_q &= \operatorname{div} \mathring{R}_q \\ \operatorname{div} u_q &= 0 \end{aligned}$$

- Spatial frequency parameter $\lambda_q = a^{(b^q)}$
- Amplitude parameter $\delta_q = \lambda_q^{-2\beta}$
- *Very heuristic* inductive assumptions:

$$\left\| \mathring{R}_q \right\|_{C_t^0 L_x^1} \leq \delta_{q+1}, \quad \|\nabla u_q\|_{C_t^0 L_x^2} \leq \delta_q^{1/2} \lambda_q, \quad \|w_{q+1}\|_{C_t^0 L_x^2} \leq \delta_{q+1}^{1/2}$$

Set-Up and Main Difficulties: The Increment

- Addition of the increment w_{q+1} shows that

$$\begin{aligned}\operatorname{div} \mathring{R}_{q+1} - \nabla(p_{q+1} - p_q) = \operatorname{div}\left(w_{q+1} \otimes w_{q+1} + \mathring{R}_q\right) \\ + (\partial_t + u_q \cdot \nabla) w_{q+1} \\ + w_{q+1} \cdot \nabla u_q\end{aligned}$$

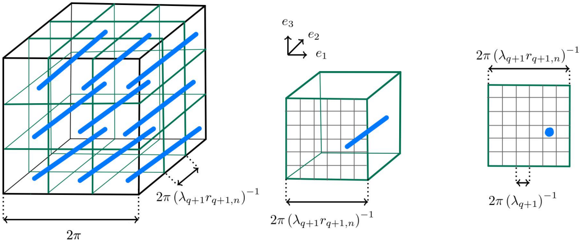
- Ansatz for w_{q+1} is

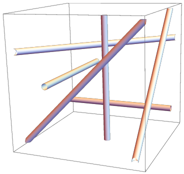
$$w_{q+1} \approx \sum_{\xi} a_\xi \left(\mathring{R}_q\right) \mathbb{W}_{\xi, r_{q+1}}$$

- a_ξ is such that

$$\sum_{\xi} a_\xi^2(\mathring{R}_q)\, \xi \otimes \xi = \rho(x,t)\,\mathrm{Id} - \mathring{R}_q$$

Set-Up and Main Difficulties: Intermittent Mikado/Pipe Flows





Set-Up and Main Difficulties: Type 1 Oscillation Error

$$\operatorname{div}\left(w_{q+1}\otimes w_{q+1}+\mathring{R}_q\right)=\operatorname{div}\left(\sum_{\xi}a_{\xi}^2(\mathring{R}_q)\mathbb{W}_{\xi,r_{q+1}}\otimes\mathbb{W}_{\xi,r_{q+1}}+\mathring{R}_q\right)$$

$$=\operatorname{div}\left(\sum_{\xi}a_{\xi}^2(\mathring{R}_q)\mathbb{P}_{=0}\left(\mathbb{W}_{\xi,r_{q+1}}\otimes\mathbb{W}_{\xi,r_{q+1}}\right)+\mathring{R}_q\right)$$

$$+\sum_{\xi}\nabla a_{\xi}^2(\mathring{R}_q)\mathbb{P}_{\neq 0}\left(\mathbb{W}_{\xi,r_{q+1}}\otimes\mathbb{W}_{\xi,r_{q+1}}\right)$$

Set-Up and Main Difficulties: The Transport Error

$$
\begin{aligned}
(\partial_t + u_q \cdot \nabla) \left(a_\xi \left(\mathring{R}_q \right) \mathbb{W}_{\xi, r_{q+1}} \right) &= (\partial_t + u_q \cdot \nabla) \left(a_\xi \left(\mathring{R}_q \right) \right) \mathbb{W}_{\xi, r_{q+1}} \\
&\quad + a_\xi \left(\mathring{R}_q \right) (\partial_t + u_q \cdot \nabla) \left(\mathbb{W}_{\xi, r_{q+1}} \right)
\end{aligned}
$$

Set-Up and Main Difficulties: Type 2 Oscillation Error

$$\operatorname{div}\left(a_\xi a_{\xi'} \mathbb{W}_{\xi, r_{q+1}}(\Phi_\xi) \otimes \mathbb{W}_{\xi', r_{q+1}}(\Phi_{\xi'})\right)$$

Heuristic Estimates: Velocity and Stress Cutoff Functions

- **Cutoffs:** We need knowledge of ∇u_q to determine a timescale for Lagrangian coordinate systems, and we need knowledge of the local size of $\mathring{R}_q$ to enact cancellation
- **Heuristic Definition:** Level set decomposition
- **Issues:** How to estimate material derivatives?

Heuristic Estimates: Three General Principles

- $\mathring{R}_q$ lives at frequency $\lambda_q = \lambda_{q+1} \cdot \frac{\lambda_q}{\lambda_{q+1}}$ and has amplitude $\delta_{q+1} = \frac{\delta_{q+1}\lambda_q}{\lambda_{q+1} \cdot \frac{\lambda_q}{\lambda_{q+1}}}$
- $\mathring{R}_{q+1}$ lives at frequency $\lambda_{q+1} = \lambda_{q+1} \cdot 1$ and has amplitude $\delta_{q+2} = \frac{\delta_{q+1}\lambda_q}{\lambda_{q+1} \cdot 1}$
- "Intermediate" errors satisfy "intermediate" estimate

Goal: Build a *sub*-perturbation $w_{q+1,n}$ which cancels $\mathring{R}_{q,n}$ and produces errors which either are small enough to be absorbed into $\mathring{R}_{q+1}$ or satisfy estimates as before but with $r_{q+1,n'} > r_{q+1,n}$.

Heuristic Estimates: 3rd General Principle (Type 2 Osc.)

$$\left\|\nabla^M \mathring{R}_{q,n}\right\|_{L^1} \leq \frac{\delta_{q+1}\lambda_q}{\lambda_{q+1} r_{q+1,n}} \left(\lambda_{q+1} r_{q+1,n}\right)^M$$

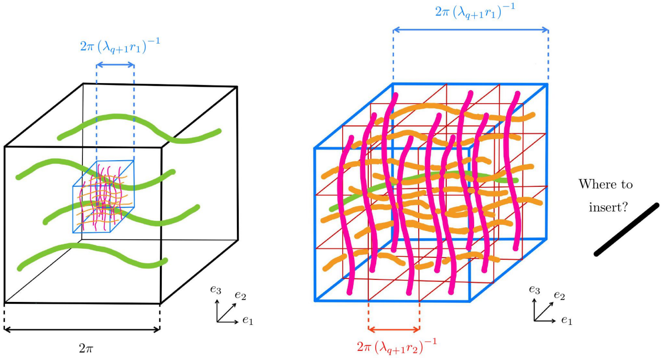
Claim: Correcting this error by pipe flows with intermittency $r_{q+1,n'}$ for $r_{q+1,n'} = {r_{q+1,n}}^{3/4}$ produces *new* Type 2 oscillation errors which vanish

Proof: Strategy - Intermittency parameter $r_{q+1,n'}$ allows for $(r_{q+1,n'})^{-2}$ disjoint placements

Step 1 - Checkerboard Cutoff Functions and "Pipe Wavelets"

Heuristic Estimates: 3rd General Principle (Type 2 Osc. cont'd)

Diagram of Stress Decomposition and Pipe Wavelets



Heuristic Estimates: 3rd General Principle (Type 2 Osc. cont'd)

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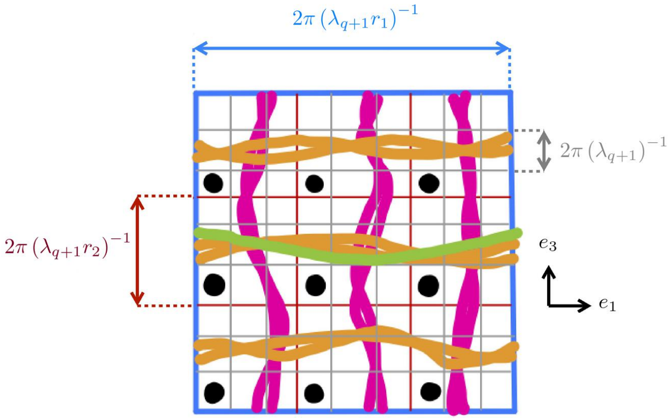
Proof: (cont'd)

Step 2 - Counting existing pipes

How many pipes which were periodized to scale $(\lambda_{q+1} r_{q+1,n'})^{-1}$ and then deformed by a Lipschitz diffeomorphism inhabit a set of diameter $(\lambda_{q+1} r_{q+1,n})^{-1}$?

Heuristic Estimates: 3rd General Principle (Type 2 Osc. cont'd)

Projection and Counting of Existing Pipes



Heuristic Estimates: 3rd General Principle (Type 2 Osc. cont'd)

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Proof: (cont'd)

Step 3 - Covering a single existing pipe

How many grid squares of size λ_{q+1}^{-1} does it take to cover a deformed pipe of thickness λ_{q+1}^{-1} which lives in a set of diameter $\left(\lambda_{q+1} r_{q+1,n}\right)^{-1}$?

Heuristic Estimates: 3rd General Principle (Type 2 Osc. cont'd)

$$\left\|\nabla^M \mathring{R}_{q,n}\right\|_{L^1} \leq \frac{\delta_{q+1}\lambda_q}{\lambda_{q+1} r_{q+1,n}} \left(\lambda_{q+1} r_{q+1,n}\right)^M$$

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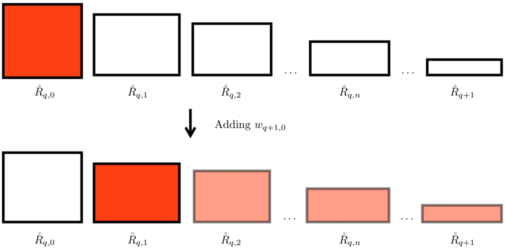
Proof: (cont'd)

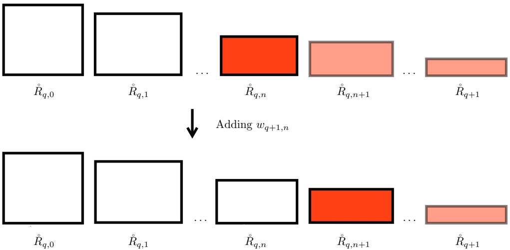
Step 4 - Pigeonhole principle and relative intermittency inequality

How many grid squares did I use to cover the existing pipes, and is this less than the number of options I have for placing the new set?

Putting Everything Together -Adding Sub-Perturbations

$w_{q+1,n}$





Thank you for your attention!

Difficulties and Solutions

1. Low-frequency, high-amplitude oscillation errors due to intermittency

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1. Low-frequency, high-amplitude oscillation errors due to intermittency
2. Sharp higher order material and spatial derivatives, and sharp local timescale information
3. Pipes from overlapping Lagrangian coordinate systems or different "bites at the apple" may intersect

1. Higher order stresses and correctors - "two (or more) bites at the apple"
2. Cutoff functions which act as a Littlewood-Paley projection in Eulerian and Lagrangian coordinates, with info about position and amplitude
3. Intermittency allows for shifts, and thus a placement technique

General Principles

Let $r_n \in [\lambda_q/\lambda_{q+1}, 1]$, and let $\mathring{R}_{q,n}$ be an error living at frequency $\lambda_{q+1} r_{q+1,n}$ and of size $\frac{\delta_{q+1}\lambda_q}{\lambda_{q+1} r_{q+1,n}}$ in L^1. $\to w_{q+1,n} = \sum_\xi a_\xi(\mathring{R}_{q,n}) \mathbb{W}_{\xi, r_{q+1,n'}}$

1. **Effect of intermittency on Type 1:** Correcting this error by a pipe flow with intermittency $r_{q+1,n'} > r_{q+1,n}$ produces a *new* Type 1 oscillation error with minimum frequency $\lambda_{q+1} r_{q+1,n'}$ and size $\frac{\delta_{q+1}\lambda_q}{\lambda_{q+1} r_{q+1,n'}}$.

- Want less intermittency!
- Goal: $r_{q+1,n'} \to 1$

$\Rightarrow$ everything ends up in $\mathring{R}_{q+1}$!

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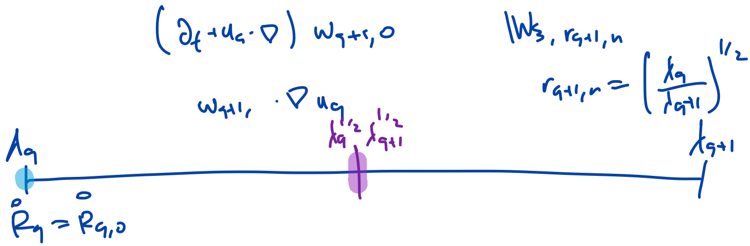
2. **Effect of intermittency on Nash/transport:** Correcting this error by a pipe flow with intermittency $r_{q+1,n'} = r_{q+1,n}^{1/2}$ produces *new* Nash and transport errors of size δ_{q+2} which are commensurate with $H^{\frac{1}{2}}$ regularity.



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3. **Effect of intermittency on Type 2:** Correcting this error by pipe flows with intermittency $r_{q+1,n'} = r_{q+1,n}{}^{3/4}$ ensures that the *new* Type 2 errors vanish.