A basis of neighbourhoods of 0 in fields of finite dp-rank (from: Will Johnson DpI, section 4)

MSRI, january 14^{th} 2021

A basis of neighbourhoods of 0

Let $\mathfrak{M}$ be a sufficiently saturated dp-finite field and $\mathcal{M}$ be a small elementary substructure of $\mathfrak{M}$.

Recall that X is heavy if it is Y-heavy for some critical set Y, namely there is $\delta \in \mathfrak{M}$ such that $\mathrm{rk}(Y \cap (X + \delta)) = \mathrm{rk}(Y)$. Furthermore, if Y' is another critical set , then X is Y'-heavy (Proposition 4.18) and we were in the middle of that proof.

A basis of neighbourhoods of 0

Let X, Y be two definable subsets of $\mathfrak{M}$ and set
$$X -_\infty Y := \{\delta \in M \colon X \cap (Y + \delta) \text{ is heavy}\}.$$

Note that $X -_\infty Y \subset X - Y$. (If $u \in X \cap (Y + \delta)$, then $u = x = y + \delta$, so $\delta = x - y$).

Candidates of basic neighbourhood of 0 inducing on $\mathfrak{M}$ a field topology: (Definition 6.3) $X -_\infty X := \{\delta \in \mathfrak{M} \colon X \cap (X + \delta) \text{ is heavy}\}$, where X is a definable *heavy* subset of $\mathfrak{M}$ (so $0 \in X_\infty X$).

When $\mathfrak{M}$ is an abelian group, not of finite Morley rank we will get two disjoint heavy sets (Theorem 5.2) (and in case $\mathfrak{M}$ is a field, a Hausdorff topology (Proposition 6.5.5)).

Properties of heavy sets

Theorem (Theorem 4.20)

1. Assume that $\mathfrak{M}$ is infinite, and X heavy, then X is infinite.
2. If $X \cup Y$ is heavy, then either X is heavy or Y is heavy.
3. If X is heavy and $X \subset Y$, then Y is heavy.
4. Let $\{D_b \colon b \in \mathfrak{M}\}$ be a definable family of subsets of $\mathfrak{M}$, then $\{b \colon D_b \text{ is heavy }\}$ is definable.
5. $X -_\infty Y$ is definable.
6. $\mathfrak{M}$ is heavy.
7. If X heavy, then for any $\alpha \in \mathfrak{M}^\times$, $\alpha \cdot X$ is heavy.
8. If X heavy, then for any $\alpha \in \mathfrak{M}$, $\alpha + X$ is heavy.
9. If either X or Y is not heavy, then $X -_\infty Y = \emptyset$.
10. If X, Y are heavy, then $X -_\infty Y$ is heavy.
11. Let X be heavy, then $0 \in X -_\infty X$.

Heaviness is well-defined

(III)

Lemma (Lemma 4.12)

Let Y be a critical set of rank ρ and Q be quasi-minimal and let $t \geq 1$ an integer. There exist pairwise distinct $q_1, \ldots, q_t \in Q$ such that

$$rk(\bigcap_{i=1}^{t}(Y + q_i)) = \rho.$$

Note that the lemma implies that $\bigcap_{i=1}^{t}(Y + q_i)$ is critical. (Indeed a translate of a critical set is critical and if a subset of a critical set has the same rank then it is also critical.)

Heaviness is well-defined

Proof (by contradiction):

So for any distinct $q_1, \ldots, q_t \in Q$, $rk(\bigcap_{i=1}^{t}(Y + q_i)) \leq \rho - 1$ (1). Let $(X_1, \ldots, X_n, P)$ be a critical configuration with target Y. By 3.23, $\rho = \sum_{i=1}^{n} rk(X_i)$. By 4.6, there exist a small model M and non-algebraic global M-invariant types p_i on X_i such that if $a \models p_1 \otimes \ldots \otimes p_n \restriction M$, then $a \in P$. Furthermore we may assume that Q is defined over M and that there is a non-algebraic M-invariant type p_0 containing Q. □

Claim (4.13)

For $k \in \mathbb{N}^*$, let
$\Omega_k := \{(a_{1,1}, \ldots, a_{1,n}, \ldots, a_{k,1}, \ldots, a_{k,n}, q_0) \in (X_1 \times \ldots \times X_n)^k \times Q$
such that

1. for each $i \in [k]$, $(a_{i,1}, \ldots, a_{i,n}) \in P$,
2. there are infinitely many $q \in Q$ such that $\bigwedge_{i=1}^{k} \left((q_0 + \sum_{j=1}^{n} a_{i,j}) \in Y + q\right)$.

Then for $k >> 0$, Ω_k is not a broad subset of $(X_1 \times \ldots \times X_n)^k \times Q$.

Note that since $\exists^\infty$ is eliminated, the sets Ω_k are definable.

Proof of Claim (by contradiction).

Let $h := rk(Q) > 0$. Choose k large enough such that $t.h + k(\rho - 1) < h + k.\rho$, equivalently $h.(t-1) < k$. By 3.23, if Ω_k were broad, $rk(\Omega_k) = h + k.\rho$. In particular Ω_k would contain a tuple of that rank (over M) (2). Let $(a_{1,1}, \ldots, a_{1,n}, \ldots, a_{k,1}, \ldots, a_{k,n}, q_0)$ be such tuple. For $i \in [k]$, let $s_i := \sum_{j=1}^{n} a_{i,j}$. By definition of Ω_k, $(a_{i,1}, \ldots, a_{i,n}) \in P$. So $s_i \in Y (= \pi(P))$. □

Proof continued.

Since the fibers of π are finite, $(a_{i,1}, \ldots, a_{i,n}) \in acl(s_i M)$. Again by definition of Ω_k, there are infinitely many $q \in Q$ such that $\{q_0 + s_1, \ldots, q_0 + s_k\} \in Y + q$. So we may choose $q_1, \ldots, q_{t-1}$ pairwise distinct and not equal to q_0 such that $q_0 + s_i \in \bigcap_{\ell=1}^{t-1} Y + q_\ell$, $i \in [k]$ (and so $q_0 + s_i \in \bigcap_{\ell=0}^{t-1} Y + q_\ell$). We have $rk(s_i/Mq_0, \ldots, q_{t-1}) = rk((a_{i,1}, \ldots, a_{i,n})/Mq_0, \ldots, q_{t-1}) \leq rk(\bigcap_{\ell=0}^{t-1} Y + q_\ell) < \rho$ (by (1)). By subadditivity of dp-rank,

$$k\rho + h \leq rk((a_{1,1}, \ldots, a_{1,n}, \ldots, a_{k,1}, \ldots, a_{k,n}, q_0, q_1, \ldots, q_{t-1})/M) \leq k(\rho - 1) + t.h,$$

contradicting (2) (recall that k has been chosen such that $k(\rho - 1) + t.h < k\rho + h$).

End of proof of the claim. □

Fix k such that $h.(t-1) < k$ and so Ω_k is not broad. Choose $(a_{1,1}, \ldots, a_{1,n}, \ldots, a_{k,1}, \ldots, a_{k,n}, q_0)$ realizing $(p_1 \otimes \ldots \otimes p_n)^{\otimes k} \otimes p_0$ over M. Let $s_i := \sum_{j=1}^{n} a_{i,j}$, $i \in [k]$. Recall that each $\bar{a}_i := (a_{i,1}, \ldots, a_{i,n}) \in P$ and so $s_i \in Y$. By Lemma 4.5, $tp(\bar{a}_i, q_0)/M)$ is broad and so $(\bar{a}_i, q_0) \notin \Omega_k$. So there are only finitely many $q \in Q$ such that $\bigwedge_{i=1}^{k} \left(q_0 + s_i \in Y + q\right)$. Since $s_i \in Y$, q_0 is among these q's, which implies that $q_0 \in acl(M, s_1 + q_0, \ldots, s_k + q_0)$. Choose ℓ minimal such that $q_0 \in acl(M, s_1 + q_0, \ldots, s_\ell + q_0)$. Note that $\ell \geq 1$, since $tp(q_0/M) = p_0$ is non-algebraic. Let $M' := M \cup \{s_1 + q_0, \ldots, s_{\ell-1} + q_0\}$. By choice of ℓ, $q_0 \notin acl(M')$; also note that $M'q_0 \subset dcl(M, q_0, (a_{i,j})_{1 \leq i \leq n, 1 \leq j \leq \ell-1})$.

We are in position to apply Lemma 4.11. Indeed, $q_0 \notin acl(M')$, $\bar{a}_\ell$ realizes the M-invariant type $p_1 \otimes \ldots \otimes p_n$ over $M'q_0$. So we can find N a small model containing M' and a N-invariant type r such that $\bar{a}_\ell q_0$ realizes $p_1 \otimes \ldots \otimes p_n \otimes r \upharpoonright N$, namely q_0 realizes $r \upharpoonright N$ (in particular r contains Q) and $\bar{a}_\ell$ realizes $p_1 \otimes \ldots \otimes p_n \upharpoonright Nq_0$. By Lemma 4.5, $tp(\bar{a}_\ell, q_0/N)$ is broad. Recall that $(X_1, \ldots, X_n, P)$ was a critical coordinate configuration $\bar{a}_\ell \in P$, Q a quasi-minimal set, $(\bar{a}_\ell, q_0) \in X_1 \times \ldots, X_n \times Q$, with a broad type over N (over which everything is defined). So by Lemma 4.10, $q_0 \notin acl(s_\ell + q_0, N)$, otherwise one contradicts the fact that the configuration is critical. However ℓ was chosen such that $q_0 \in acl(M, s_1 + q_0, \ldots, s_\ell + q_0) \subset acl(M', s_\ell + q_0)$, a contradiction.

Lemma (Proposition 4.14)

Let Y be a critical set and $Q_1, \ldots, Q_n$ be quasi-minimal. Then for every m there exist $\{q_{ij}\}_{i\in[n],j\in[m]}$ such that

1. for fixed $i \in [n]$, $q_{i,1}, \ldots, q_{i,m}$ consist of m distinct elements of Q_i
2. the intersection $\bigcap_{\eta:[n]\to[m]}(Y + \sum_{i=1}^{n} q_{i,\eta(i)})$ is critical.

Proof.

We proceed by induction on n. The case $n = 1$ is Lemma 4.12. Assume $n > 1$, so by induction we may find $\{q_{ij}\}_{i\in[n-1],j\in[m]}$ with for fixed $i \in [n-1]$, $q_{i,1}, \ldots, q_{i,m}$ consist of m distinct elements of Q_i and $Y' := \bigcap_{\eta:[n-1]\to[m]}(Y + \sum_{i=1}^{n-1} q_{i,\eta(i)})$ is critical. By the preceding lemma, there are pairwise distinct elements $q_{n,1}, \ldots, q_{n,m} \in Q_n$ such that $rk(\bigcap_{j=1}^{m}(Y' + q_{n,j})) = rk(Y')$ (and so is critical). Set $Y'' := (\bigcap_{j=1}^{m}(Y' + q_{n,j})$. Unravelling what is Y' we get the result. □

Indeed, $x \in Y''$ iff for every $j \in [m]$, and every $\eta : [n-1] \to [m]$, $x - q_{n,j} - q_{n-1,\eta(n-1)} - \ldots - q_{1,\eta(1)} \in Y$. So $Y'' = \bigcap_{\eta:[n]\to[m]} (Y + \sum_{i=1}^{n} q_{i,\eta(i)})$.

Lemma (Corollary 4.15)

Let Y be a critical set and $Q_1, \ldots, Q_n$ be quasi-minimal. There exists $\delta \in \mathfrak{M}$ such that

$$\{(x_1, \ldots, x_n) \in Q_1 \times \ldots \times Q_n \colon x_1 + \ldots + x_n \in Y + \delta\}$$

is a broad subset of $Q_1, \ldots, Q_n$

Proof.

By Proposition 4.14 applied to the quasi-minimal sets $(-Q_i)$, we can find for every m, $\{q_{ij}\}_{i\in[n],j\in[m]}$ such that

1. for fixed $i \in [n]$, $q_{i,1}, \ldots, q_{i,m}$ consist of m-distinct elements of Q_i
2. the intersection $\bigcap_{\eta:[n]\to[m]}(Y - \sum_{i=1}^{n} q_{i,\eta(i)})$ is critical. (and in particular non-empty)

Let $-\delta \in \bigcap_{\eta:[n]\to[m]}(Y - \sum_{i=1}^{n} q_{i,\eta(i)})$. So for any $\eta : [n] \to [m]$, $-\delta \in Y - \sum_{i=1}^{n} q_{i,\eta(i)}$. Equivalently, $\sum_{i=1}^{n} q_{i,\eta(i)} \in Y + \delta$. By compactness, we can find, for each $i \in [n]$, $(q_{i,j})_{j\in\mathbb{N}}$ pairwise distinct elements of $Q_i(\mathfrak{M})$ such that for any $\eta : [n] \to \mathbb{N}$, $\sum_{i=1}^{n} q_{i,\eta(i)} \in Y + \delta$. This means that $\{(q_1, \ldots, q_n) \in Q_1 \times \ldots \times Q_n : q_1 + \ldots + q_n \in Y + \delta\}$ is a broad subset of $Q_1 \times \ldots \times Q_n$. □

Proof of Proposition 4.18

Let Y, Y' be two critical sets and let X be a definable subset of $\mathfrak{M}$. Assume that X is Y-heavy.

Proof.

Let δ_0 be such that $rk(Y \cap (X + \delta_0)) = rk(Y)$. We have to show that X is Y'-heavy. Note that $X' := Y \cap (X + \delta_0)$ is critical as a subset of a critical set of the same rank. First we show that X' is Y'-heavy. Since Y' is a critical set, we have $(A_1, \ldots, A_n, P)$ a critical configuration with target Y'. By Theorem 3.10, there exist infinite definable subsets $D_i \subset A_i$ such that $D_1 \times \ldots \times D_n \setminus P$ is an hyperplane, namely for every $b \in D_n$

$$\{(d_1, \ldots, d_{n-1}) \in D_1 \times \ldots \times D_{n-1} : (d_1, \ldots, d_{n-1}, b) \notin P\}$$

is not a broad subset of $A_1 \times \ldots \times A_{n-1}$ $(\star)$. □

Proof.

By Corollary 4.15 (the D_i's are quasi-minimal and X' is critical, there exists $\delta_1 \in \mathfrak{M}$ such that

$$\{(x_1, \ldots, x_n) \in D_1 \times \ldots \times D_n \colon x_1 + \ldots + x_n \in X' + \delta_1\}$$

is a broad subset of $D_1 \times \ldots \times D_n$ (and so of $A_1 \times \ldots \times A_n$). By $(\star)$, $\{(x_1, \ldots, x_n) \in P \colon x_1 + \ldots + x_n \in X' + \delta_1\}$ is a broad subset of $A_1 \times \ldots \times A_n$. □

Proof.

Recall that the map $\pi : (x_1, \ldots, x_n) \mapsto x_1 + \ldots + x_n$ has finite fibers on P. So the subset of Y': $\{x_1 + \ldots + x_n :$

$$(x_1, \ldots, x_n) \in P, x_1 + \ldots + x_n \in X' + \delta_1\} = Y' \cap (X' + \delta_1)\}$$

has full rank. So X' is Y'-heavy. Finally $X' \subset X + \delta_0$, so $Y' \cap (X + \delta_0 + \delta_1)$ has full rank in Y' and so X is Y'-heavy.

□