A basis of neighbourhoods of 0 in fields of finite dp-rank (from: Will Johnson DpI, section 4)

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Heaviness is well-defined

(IV)

Let $\mathfrak{M}$ be a sufficiently saturated dp-finite field and $\mathcal{M}$ be a small elementary substructure of $\mathfrak{M}$.

Recall that X is heavy if it is Y-heavy for some critical set Y, namely there is $\delta \in \mathfrak{M}$ such that rk$(Y \cap (X + \delta)) =$rk(Y).

Furthermore, if Y' is another critical set , then X is Y'-heavy (Proposition 4.18).

Heaviness is well-defined

Lemma (Proposition 4.14)

Let Y be a critical set and $Q_1, \ldots, Q_n$ be quasi-minimal. Then for every m there exist $\{q_{ij}\}_{i \in [n], j \in [m]}$ such that

1. for fixed $i \in [n]$, $q_{i,1}, \ldots, q_{i,m}$ consist of m distinct elements of Q_i
2. the intersection $\bigcap_{\eta : [n] \to [m]} (Y + \sum_{i=1}^{n} q_{i,\eta(i)})$ is critical.

Lemma (Corollary 4.15)

Let Y be a critical set and $Q_1, \ldots, Q_n$ be quasi-minimal. There exists $\delta \in \mathfrak{M}$ such that

$$\{(x_1, \ldots, x_n) \in Q_1 \times \ldots \times Q_n : x_1 + \ldots + x_n \in Y + \delta\}$$

is a broad subset of $Q_1, \ldots, Q_n$

Proof of Corollary 4.15

Proof.

By Proposition 4.14 applied to the quasi-minimal sets $(-Q_i)$, we can find for every m, $\{q_{ij}\}_{i\in[n],j\in[m]}$ such that

1. for fixed $i \in [n]$, $q_{i,1}, \ldots, q_{i,m}$ consist of m-distinct elements of Q_i
2. the intersection $\bigcap_{\eta:[n]\to[m]}(Y - \sum_{i=1}^{n} q_{i,\eta(i)})$ is critical (and in particular non-empty)

Let $-\delta \in \bigcap_{\eta:[n]\to[m]}(Y - \sum_{i=1}^{n} q_{i,\eta(i)})$. So for any $\eta : [n] \to [m]$, $-\delta \in Y - \sum_{i=1}^{n} q_{i,\eta(i)}$. Equivalently, $\sum_{i=1}^{n} q_{i,\eta(i)} \in Y + \delta$. By compactness, we can find, for each $i \in [n]$, $(q_{i,j})_{j\in\mathbb{N}}$ pairwise distinct elements of $Q_i(\mathfrak{M})$ such that for any $\eta : [n] \to \mathbb{N}$, $\sum_{i=1}^{n} q_{i,\eta(i)} \in Y + \delta$. This means that $\{(q_1, \ldots, q_n) \in Q_1 \times \ldots \times Q_n : q_1 + \ldots + q_n \in Y + \delta\}$ is a broad subset of $Q_1 \times \ldots \times Q_n$. □

Let X, Y be two definable subsets of $\mathfrak{M}$ and set

$$X -_{\infty} Y := \{\delta \in M \colon X \cap (Y + \delta) \text{ is heavy}\}.$$

Note that $X -_{\infty} Y \subset X - Y$. (If $u \in X \cap (Y + \delta)$, then $u = x = y + \delta$, so $\delta = x - y$).

Properties of heavy sets

Theorem (Theorem 4.20)

1. Assume that $\mathfrak{M}$ is infinite, and X heavy, then X is infinite.
2. If $X \cup Y$ is heavy, then either X is heavy or Y is heavy.
3. If X is heavy and $X \subset Y$, then Y is heavy.
4. Let $\{D_b \colon b \in \mathfrak{M}\}$ be a definable family of subsets of $\mathfrak{M}$, then $\{b \colon D_b \text{ is heavy }\}$ is definable.
5. $X -_\infty Y$ is definable.
6. $\mathfrak{M}$ is heavy.
7. If X heavy, then for any $\alpha \in \mathfrak{M}^\times$, $\alpha \cdot X$ is heavy.
8. If X heavy, then for any $\alpha \in \mathfrak{M}$, $\alpha + X$ is heavy.
9. If either X or Y is not heavy, then $X -_\infty Y = \emptyset$.
10. If X, Y are heavy, then $X -_\infty Y$ is heavy.
11. Let X be heavy, then $0 \in X -_\infty X$.

Proof of 4.20

In theorem above, some properties (in the text) are stated for light sets, but a light set is the complement of an heavy set!

1. Assume that $\mathcal{M}$ is infinite, and X heavy, then X is infinite.
   The critical rank is always > 0 and since a heavy set contains a critical set, it is of rank > 0 and so infinite.
2. If $X \cup Y$ is heavy, then either X is heavy or Y is heavy.
   Assume $X \cup Y$ is Z-heavy for some critical set Z. Then there $\delta \in M$ such that $rk((X \cup Y) + \delta) \cap Z) = rk(Z)$. But $(X \cup Y) + \delta = (X + \delta) \cup (Y + \delta)$ and $((X + \delta) \cup (Y + \delta)) \cap Z = ((X + \delta) \cap Z) \cup ((Y + \delta) \cap Z)$.
3. If X is heavy and $X \subset Y$, then Y is heavy.
   $(X + \delta) \cap Z \subset (Y + \delta) \cap Z$ and so if $(X + \delta) \cap Z$ is a critical set, then so is $(Y + \delta) \cap Z$.

Proof of 4.20 (continued)

1. Let $\{D_b \colon b \in \mathfrak{M}\}$ be a definable family of subsets of $\mathfrak{M}$, then $\{b \colon D_b \text{ is heavy }\}$ is definable.
   By Proposition 4.18, we may assume that the sets D_b are Y-heavy for the same Y. Namely that $\mathrm{rk}(Y \cap (D_b + \delta_b)) = \mathrm{rk}(Y)$. But by Proposition 4.3(1), full rank is definable, namely $\{b \colon rk(Y \cap (D_b + \delta_b)) = rk(Y)\}$ is definable.
2. The set $X -_\infty Y$, where X and Y are definable, is definable (and $X -_\infty Y \subseteq X - Y$).
   $X -_\infty Y = \{b \in \mathfrak{M} \colon (X \cap (Y + b)) \text{ is heavy}\}$.
3. M is heavy.
   M contains a critical set.

Proof of 4.20 (continued)

1. If X heavy, then for any $\alpha \in M^{\times}$, $\alpha \cdot X$ is heavy.
   The map $x \mapsto \alpha x$ is a definable bijection.
2. If X heavy, then for any $\alpha \in M$, $\alpha + X$ is heavy.
   the map $x \mapsto x + \alpha$ is a definable bijection.
3. If either X or Y is not heavy, then $X -_{\infty} Y = \emptyset$.
   Suppose otherwise. Let $\delta \in X -_{\infty} Y$. So $X \cap (Y + \delta)$ is heavy. But this set is included in X and $Y + \delta$, so X and $Y + \delta$ are heavy. If $Y + \delta$ is heavy, Y is heavy.

Proof of 4.20 (continued)

1. If X, Y are heavy, then $X -_\infty Y$ is heavy.
   W.l.o.g. we may assume that X and Y are critical sets (translates of critical sets are critical and if you contain a critical set, you are critical). Let $(A_1, \ldots, A_n, P)$ be a coordinate configuration for Y. By Theorem 3.10 (and the beginning of the proof), there are infinite definable subsets $Q_i \subset A_i$ (so quasi-minimal) such that $(Q_1 \times \ldots \times Q_n) \setminus P$ is not broad. By Corollary 4.15, there is $\delta \in \mathfrak{M}$ such that

   $$\{(q_1, \ldots, q_n, q_1', \ldots, q_n') \in (Q_1 \times \ldots \times Q_n)^2 : \\ q_1 + \ldots + q_n + q_1' + \ldots + q_n' \in X + \delta\}$$

   is broad as a subset of $(Q_1 \times \ldots \times Q_n)^2$. So we can find $(q_{i,j}), (q'_{i,j}) \in Q_i$, $1 \leq i \leq n$, $j \in \mathbb{N}$, with $q_{i,j} \neq q_{i,k}$, $q'_{i,j} \neq q'_{i,k}$, for $j \neq k$ and for any η, $\eta' : [n] \to \mathbb{N}$,

   $$\sum_{i=1}^{n} q_{i,\eta(i)} + q'_{i,\eta'(i)} \in X + \delta.$$

Let $s_\eta := \sum_{i=1}^{n} q_{i,\eta(i)}$, then the elements (q'_{ij}) witness that the set

$$\{(q'_1, \ldots, q'_n) \in (Q_1 \times \ldots \times Q_n) \colon s_\eta + q'_1 + \ldots + q'_n \in X + \delta\}$$

is a broad subset of $Q_1 \times \ldots \times Q_n$. Since $(Q_1 \times \ldots \times Q_n \setminus P)$ is not broad, it follows that
$W := \{(q'_1, \ldots, q'_n) \in P \colon s_\eta + q'_1 + \ldots + q'_n \in X + \delta\}$ is a broad subset of $Q_1 \times \ldots \times Q_n$, as well.
By 3.23, $rk(W) = \sum_{i=1}^{n} rk(Q_i) = \sum_{i=1}^{n} rk(A_i) =: \rho$. Since $(A_1, \ldots, A_n, P)$ is a coordinate configuration for Y, we have $q'_1 + \ldots + q'_n \in Y$ (and the fibers of this map are finite) $(\star)$. So $(s_\eta + Y) \cap (X + \delta)$ has rank ρ. The translate $(s_\eta + Y - \delta) \cap X$ has rank ρ (and since its intersection with X is itself), it is X-heavy. So, $s_\eta - \delta \in X -_\infty Y$.
This holds for any function η, so
$\{(q_1, \ldots, q_n) \in (Q_1 \times \ldots \times Q_n) \colon q_1 + \ldots + q_n - \delta \in X -_\infty Y\}$ is broad as a subset of $Q_1 \times \ldots \times Q_n$. So by 3.23, it has full rank ρ. Again (see $(\star)$), this implies that $(Y - \delta) \cap (X -_\infty Y)$ has rank ρ. So $X -_\infty Y$ is heavy. $\square$

A basis of neighbourhoods of 0

Candidates of basic neighbourhood of 0 inducing on $\mathfrak{M}$ a field topology: $X -_\infty X := \{\delta \in \mathfrak{M} \colon X \cap (X + \delta) \text{ is heavy}\}$, where X is a definable *heavy* subset of $\mathfrak{M}$ (so $0 \in X -_\infty X$)–(Definition 6.3).

We have seen that if X is definable, then $X -_\infty X$ is definable and if it is not empty X should be heavy (and if X is heavy, then $X -_\infty X$ is heavy).

Basic neighbourhoods

Remark (Remark 6.2)

1. Let $\delta_1, \delta_2 \in \mathfrak{M}$, then
   $(X + \delta_1) -_\infty (Y + \delta_2) = X -_\infty Y + (\delta_1 - \delta_2)$.
   Translates of heavy sets are heavy.
2. For any $\alpha \in \mathfrak{M}^\times$, $\alpha \cdot X -_\infty \alpha \cdot Y = \alpha \cdot (X -_\infty Y)$. So, If U is a basic neighbourhood and $\alpha \in \mathfrak{M}^\times$, then $\alpha \cdot U$ is a basic neighbourhood.
   If Z is heavy, $\alpha.Z$ is heavy.
3. If $X' \subset X$ and $Y' \subset Y$, then $X' -_\infty Y' \subseteq X -_\infty Y$.

Basic neighbourhoods

Remark (Remark 6.4)

If $\mathcal{M}$ is a small model, then every M-definable basic neighbourhood is of the form $X -_\infty X$, where X is heavy and M-definable.

Let U be a basic neighbourhood of the form $X -_\infty X$ and let $\varphi(x; y)$ be such that $X = \varphi(\mathfrak{M}; b)$, $b \in \mathfrak{M}$. Since the set of parameters c such that $\varphi(\mathfrak{M}; c)$ is heavy is definable and since U is M-definable, $\{c \in \mathfrak{M} \colon U = \varphi(\mathfrak{M}, c) -_\infty \varphi(\mathfrak{M}, c)$ and $\varphi(\mathfrak{M}, c)$ is heavy $\}$ is definable, M-invariant, non-empty (it contains b) and so it is M-definable. So it has a non-empty intersection with M. Let b' in that intersection and let $X' := \varphi(\mathfrak{M}, b')$. We have $U = \varphi(\mathfrak{M}, b') -_\infty \varphi(\mathfrak{M}, b')$ with $\varphi(\mathfrak{M}, b')$ heavy, $b' \in M$.

Basic neighbourhoods

Proposition (Proposition 6.5)

1. If U is a basic neighbourhood, then U is heavy.
2. If U is a basic neighbourhood, then $0 \in U$.
3. If U is a basic neighbourhood and $\alpha \in \mathfrak{M}^\times$, then $\alpha \cdot U$ is a basic neighbourhood.
4. If U_1, U_2 are basic neighbourhoods, then there is a basic neighbourhood $U_3 \subset U_1 \cap U_2$. If U_1, U_2 are M-definable, then we can choose U_3 to be M-definable.
5. If $\mathfrak{M}$ is not of finite Morley rank and $a \in \mathfrak{M} \setminus \{0\}$, then there is a basic neighbourhood U with $a \notin U$. If a is M-definable, we may choose U to be M-definable.

Proof of Proposition 6.5 (3)

If U_1, U_2 are basic neighbourhoods, then there is a basic neighbourhood $U_3 \subset U_1 \cap U_2$. If U_1, U_2 are M-definable, then we can choose U_3 to be M-definable.

Proof.

Set $U_i := X_i -_\infty X_i$, $i = 1, 2$, with X_i heavy. By Theorem 4.20, $X_1 -_\infty X_2$ is heavy (so non empty). Let $\delta \in X_1 -_\infty X_2$. Set $X_3 := X_1 \cap (X_2 + \delta)$. By choice of δ, X_3 is heavy. Check that $X_3 -_\infty X_3 \subseteq U_1 \cap U_2$. Let $\varphi(x; y)$ and $b \in \mathfrak{M}$ be such that $X_3 = \varphi(\mathfrak{M}, b)$. Consider $\{c \colon \emptyset \neq \varphi(\mathfrak{M}, c) -_\infty \varphi(\mathfrak{M}, c) \subset U_1 \cap U_2\}$. (Note that since $\emptyset \neq \varphi(\mathfrak{M}, c) -_\infty \varphi(\mathfrak{M}, c)$, $\varphi(\mathfrak{M}, c)$ is heavy). It is definable, contains b and M-invariant. So it is M-definable and so we can choose $b_0 \in M$ such that $\emptyset \neq \varphi(\mathfrak{M}, b_0) -_\infty \varphi(\mathfrak{M}, b_0) \subset U_1 \cap U_2$ and $\varphi(\mathfrak{M}, b_0)$ heavy. □

Proof of Proposition 6.5 (4)

Since $\mathfrak{M}$ is not of finite Morley rank, there are two disjoint heavy sets: X, Y (Theorem 5.2). Since $X -_\infty Y$ is heavy, it is not empty. Let $\delta \in X -_\infty Y$. Consider $X' := X \cap (Y + \delta) \subseteq X$: it is heavy by choice of δ, as well as $Y' := X' - \delta = (X - \delta) \cap Y \subseteq Y$. So $X' \cap Y' = \emptyset$. This implies that $\delta \notin Y' -_\infty Y'$ $(Y' \cap (Y' + \delta) = Y' \cap X')$. Since Y' is heavy, $U_0 := Y' -_\infty Y'$ is a neighbourhood of 0, which does not contain δ. In particular $\delta \neq 0$.

Since $\mathfrak{M}$ is a field, given any $\alpha \neq 0$, there is $\beta \in \mathfrak{M}$ such that $\alpha = \beta\delta$. So βU_0 is again a basic neighbourhood of 0 which does not contain α.

Finally if $\alpha \in M \setminus \{0\}$. We have shown that $\alpha \notin \varphi(\mathfrak{M}, b) -_\infty \varphi(\mathfrak{M}, b)$, with $\varphi(\mathfrak{M}, b)$ heavy. Consider $\{b \in \mathfrak{M} : \varphi(\mathfrak{M}, b)$ heavy and $\alpha \notin \varphi(\mathfrak{M}, b) -_\infty \varphi(\mathfrak{M}, b)\}$. It is definable, M-invariant and not empty. So, it is M-definable. So we can find such $b \in M$, which produces a basic M-definable neighbourhood of 0 not containing α.

Externally definable sets

Lemma (Lemma 4.21)

Let $\mathcal{M}$ a small model defining a critical coordinate configuration. Let Z be an M-definable heavy set. Let $D_1, \ldots, D_m$ be $\mathfrak{M}$-definable sets such that $Z(M) \subseteq D_1 \cup \ldots \cup D_m$. Then there is $Z' \subset Z$, $1 \leq j \leq m$ such that Z' is M-definable, heavy and $Z'(M) \subset D_j$.

Proof.

Let Y be a critical set such that Z is Y-heavy. Let $(X_1, \ldots, X_n, P)$ be a critical M-definable configuration with target Y. Let $\delta \in \mathfrak{M}$ be such that $Y \cap (Z + \delta)$ has full rank in Y. W.l.o.g. $\delta \in M$. Replacing Z by $Z + \delta$ and D_i by $D_i + \delta$, we may assume that $\delta = 0$. Consider $R := \{(x_1, \ldots, x_n) \in P \colon x_1 + \ldots + x_n \in Z\}$. This set is broad since $Y \cap Z$ has full rank in Y by Theorem 3.23. □

Proof.

Note that $R(M)$ is a union of externally definable sets: $R(M) = \bigcup_{i=1}^{m}\{(x_1,\dots,x_n) \in P(M) \colon x_1 + \dots + x_n \in Z \cap D_i\}$. By Lemma 3.13 on externally definable sets, there is a M-definable subset R' of R and $j \in [m]$ such that R' is broad and $(x_1,\dots,x_n) \in R'(M) \Rightarrow x_1 + \dots + x_n \in D_j$. Define $Z' := \{u \colon \exists x_1,\dots,\exists x_n \; u = x_1 + \dots + x_n \;\&\; (x_1,\dots,x_n) \in R'\}$. Then Z' is critical, Z' is M-definable, $Z' \subset Y \cap Z$, and Z' is Y-heavy. Let us show that $Z'(M) \subset D_j$. Indeed given $u \in Z'(M)$ there exist finitely many tuples $(x_1,\dots,x_n) \in \mathfrak{M}$ such that $u = x_1 + \dots + x_n$. Since $\mathcal{M} \preceq \mathfrak{M}$, they belong to M and so to $R'(M)$. So $Z'(M) \subset D_j$.

□

Lemma (Lemma 4.22)

Let $\mathcal{M}$ a small model defining a critical coordinate configuration. Let Z be an M-definable heavy set. Let W be $\mathfrak{M}$-definable with the property that $Z(M) \subset W$. Then W is heavy.

Proof.

Assume that $Z$is Y-heavy and let $(X_1, \ldots, X_n, P)$ be a critical M-definable configuration with target Y. By using the same trick as in the beginning of the proof of Lemma 4.21, we may assume that $Y \cap Z$ has full rank in Y. So, the M-definable set $\{(x_1, \ldots, x_n) \in P \colon x_1 + \ldots + x_n \in Z\}$ is broad in $X_1 \times \ldots \times X_n$ by 3.23. By Lemma 3.14 (on externally definable sets), $\{(x_1, \ldots, x_n) \in P \colon x_1 + \ldots + x_n \in W\}$ is broad. By 3.23, $rk(W \cap Y) = rk(Y)$, so W is Y-heavy. □

Infinitesimals

Definition (Definition 6.6)

Let $\mathcal{M}$ be a small model. An element $\epsilon \in \mathfrak{M}$ is M-infinitesimal if ϵ belongs to every M-definable basic neighbourhood, equivalently if for every M-definable heavy set $X \cap (X + \epsilon)$ is heavy.

Denote the set of M-infinitesimals by I_M.

The set I_M is type-definable over M as

$$\bigcap_{X \text{ heavy and M}-\text{definable}} X -_\infty X.$$