Where Section 8 sits in the overall scheme of things.

In Section 1, page 6, mention is made of a lattice of posets (of linear subspaces and/or abelian subgroups) where join is given by sum $A \vee B = A + B$ and meet is given by $A \wedge B = (A \cap B)^{00}$.

It is also noted that it is desirable for meet to be given just by $A \cap B$, as would be the case if $(A \cap B)^{00} = A \cap B$.

The work of section 8 gives a recipe for guaranteeing, for J in a certain family of subgroups, that $J^{00} = J$.

(I_M as discussed earlier is an example of such.)

In Section 10, Proposition 10.1, the results of Section 8 are applied to a certain poset of linear subspaces of the Monster Model $\mathbb{M}$ of a finite dp-rank expansion of a field, assuming that $\mathbb{M}$ is not of finite Morley rank.

And a couple of reminders about G^{00} before we start:

For any type-definable group G in the NIP setting, the minimal type definable subgroup G^{00} of G is also type-definable, and the index $|G/G^{00}|$ is bounded.

Moreover, if G is $\omega-$definable, i.e., is definable over a countable set, then so is G^{00}.

Now for Section 8:

Throughout this section $(G, +, ...)$ is a monster model abelian group of finite dp-rank n. G may have additional structure as well, e.g. as a group inside a field $\mathbb{M}$ of finite dp-rank.

Our starting point is the proof of a result about NTP_2 structures in Chernikov-Kaplan-Simon (2015, Proposition 4.5), which in turn goes back to Kaplan-Shelah (2011). This proposition tells us you can't dig too deeply, in terms of finite burden and inp patterns; the proof works for finite dp-rank.

Fact 8.1: Let G be as above, of dp-rank n and let $G_0, ..., G_n$ be type-definable subgroups of G. Then for some $k, 0 \leq k \leq n$,

$$(G_0 \cap ... \cap G_n)^{00} = (G_0 \cap ... G_{k-1} \cap G_{k+1} \cap ... \cap G_n)^{00}$$

Our goal is to obtain uniform bounds, depending only on the group G, for the indices of type-definable subgroups, provided that these indices are bounded. But we start by considering all the type-definable subgroups containing a fixed one.

Using Erdos-Rado

Lemma 8.3 states that for any cardinal κ there is a cardinal $\tau = \tau(\kappa)$ such that for any family $\{H_\alpha\}_{\alpha<\tau}$ of type-definable subgroups of G there are subsets S_1, S_2 of τ such that $S_1 = \{i_1, ... i_{2n}\}$ is finite and $|S_2| = \kappa$, with

$(H_{i_1} \cap H_{i_2} \cap ... \cap H_{i_{2n}})^{00}$ contained in H_α for all α in S_2.

Given a collection of H_α's as above, with κ infinite, and given any $\alpha_1 < ... < \alpha_{n+1}$, use 8.1 to choose k least in $\{1, ..., n+1\}$ such that
$(H_{\alpha_1} \cap ... \cap H_{\alpha_{n+1}})^{00}$ can be attained omitting H_{α_k}.

This partitions the ordered $n+1$ tuples from an infinite set of subscripts into $n+1$ sets, so we find a τ sufficiently large that a homogeneous subset of size at least $(\kappa)^+$ exists, which is to say that a fixed k works for every ordered $n+1$ tuple from the homogeneous set.
This guarantees that for any $\alpha_1 < ... < \alpha_{n+1} < (\kappa)^+$ we get that H_{α_k} contains $(H_{\alpha_1} \cap ... \cap H_{\alpha_{n+1}})^{00} = (H_{\alpha_1} \cap ... \cap H_{\alpha_{k-1}} \cap H_{\alpha_{k+1}} \cap ... \cap H_{\alpha_{n+1}})^{00}$. Let $S_1 = \{1, 2, ..., n, \kappa+1, ..., \kappa+n\}$ and let S_2 be the interval $[n+1, \kappa]$.

Lemma 8.2. Let H be a type-definable subgroup of G. There is a cardinal $\kappa = \kappa(G, H)$ such that for any type-definable subgroup H' with $H < H' < G$

either

H'/H is unbounded (over elementary extensions of G),

or

$|H'/H| \leq \kappa$ for all elementary extensions of G.

Proof of 8.2

Assume H is type-definable over the empty set. Choose κ using Morley-Erdos-Rado so that

(*) for any sequence $\{a_\alpha\}_{\alpha<\kappa}$ from G there is a countable 0-indiscernible subsequence $\{b_i\}_{i\in\mathbb{N}}$ such that
for any $i_1 < ... < i_n$ there exist $\alpha_1 < ... < \alpha_n$
with $a_{\alpha_1}, ..., a_{\alpha_n}$ elementarily equivalent to $b_{i_1}, ..., b_{i_n}$.

Now suppose there is H' such that $|H'/H| \geq \kappa$. If $|H'/H|$ is unbounded, fine. If not, suppose for some $\lambda \geq \kappa$ we have $|H'/H| < \lambda$ in all elementary extensions.

Choose a sequence $\{a_\alpha\}_{\alpha<\kappa}$ from H' of distinct coset representatives over H. Apply (*) to get $\{b_i\}_{i\in\mathbb{N}}$ as above. In particular the b_i's will lie in distinct H-cosets. By indiscernibility of the b_i's there is a 0-definable set D containing H such that $b_i - b_j$ is not in D whenever $i \neq j$.

But now we can create a consistent type in variables x_α for $\alpha < \lambda$ which mimics the behavior of the $\alpha_1 < ... < \alpha_n$ above and also requires that all x_α lie in H'. A realization of this type would provide distinct H-coset representatives in H', too many (at least λ many).

Now we come to the main theorem of this section.
Recall our standing assumption, that G is a monster model abelian group, perhaps with extra structure, and that we assume G is dp-finite.

Theorem 8.4 There is a cardinal $\kappa = \kappa(G)$ such that for any type-definable subgroup $H < G$ the index of H^{00} in H is less than κ. The same is true for all elementary extensions of G.

Proof:

Up to automorphism there is a bounded number of ω-definable subgroups of G. Thus by Lemma 8.2, there is a cardinal κ_0 such that for any ω-definable group K and any type-definable group K' containing K,
either $|K'/K| < \kappa_0$ or $|K'/K|$ is unbounded.

Claim 8.5 If H is a type-definable group and K is an ω-definable group containing H^{00} then $|H/H \cap K| < \kappa_0$.

Proof of claim 8.5. $H/H \cap K$ is bounded, since $H^{00} \subset H \cap K \subset H$. But $H/H \cap K$ cannot be too big, since it is isomorphic to

$(H+K)/K$. Now that we know it is bounded then it has cardinality $< \kappa_0$.

We are now ready to choose κ. Let $\kappa_1 = \tau((2^{\kappa_0})^+)$, where τ is given by Lemma 8.3, and let $\kappa = (\kappa_0)^{\kappa_1}$.

Claim 8.6 If H is a type-definable subgroup of G then there are fewer than κ_1 subgroups of the form $H \cap K$ where K is ω-definable and K contains H^{00}.

Proof of claim 8.6. If not let $\{K_\alpha\}$ be given for all $\alpha < \kappa_1$ such that each contains H^{00} and such that the $\{K_\alpha\}$'s have pairwise distinct intersections with H. Applying Lemma 8.3 to the family of K_α's we get a finite set S_1 and $|S_2| = (2^{\kappa_0})^+$ so that $(K_{i_1} \cap K_{i_2} \cap ... \cap K_{i_{2n}})^{00}$ is contained in K_α for all α in S_2.

The lefthand side of the above equation is an ω-definable group J containing H^{00}, which means $|H/H \cap J| < \kappa_0$ by Claim 8.5. But there are too few possibilities for pairwise distinct $H \cap K_\alpha$ between H and J, given that $|S_2| > 2^{\kappa_0}$. This proves the claim.

Thus we have κ as a bound for the index of H^{00} in H. This means H^{00} is an intersection of fewer than κ_1 subgroups, indeed fewer than κ_1 ω-definable groups. This puts H/H^{00} injectively into a product of fewer than κ_1 of the $H/H \cap K$'s. This set has size at most κ.

Now how does all this get used? In section 10, in the following form:

Corollary 8.7. Let $\mathbb{M}$ be a field of finite dp-rank. There is a cardinal κ such that for any small elementary submodel M of $\mathbb{M}$ such that M is not too small, namely has size at least κ, and for any J a type-definable linear subspace of $\mathbb{M}$, we have $J = J^{00}$.

Notice that J is only assumed to be a type-definable subspace of $\mathbb{M}$, not necessarily of M.